Some validated symbolic-numeric approximation algorithms

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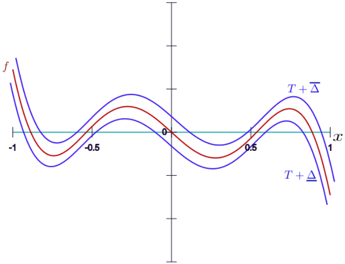
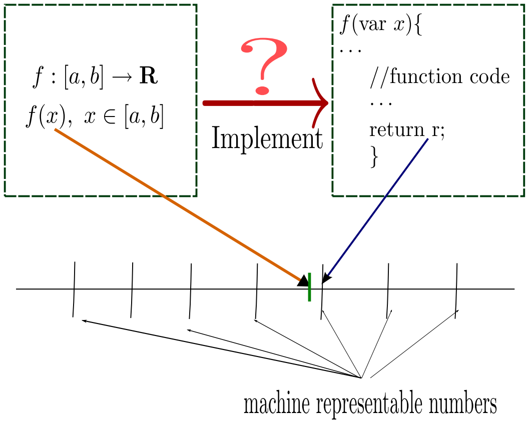


Winter Workshop on Dynamics, Topology and Computations, BEDLEWO, Poland

January 28 - February 3, 2018

Efficient Machine Implementation of Correctly Rounded Elementary Functions



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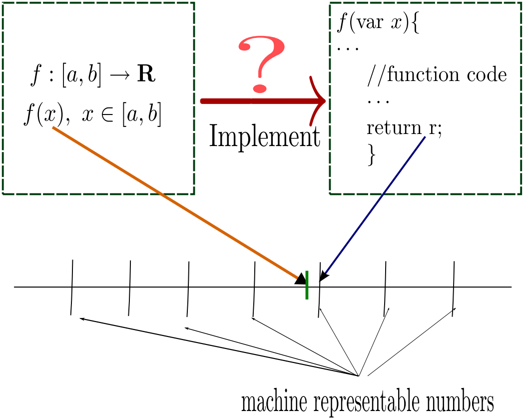

- Numerically compute best polynomial approximation p w.r.t $|| \cdot ||_\infty$.

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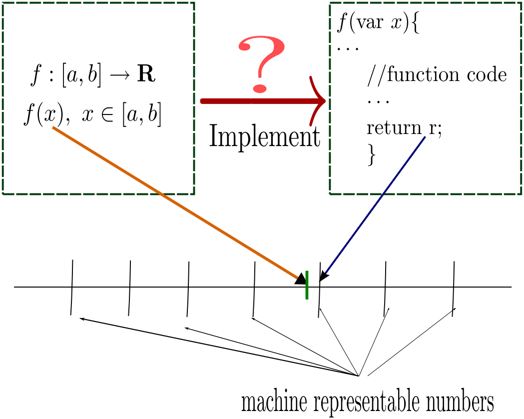
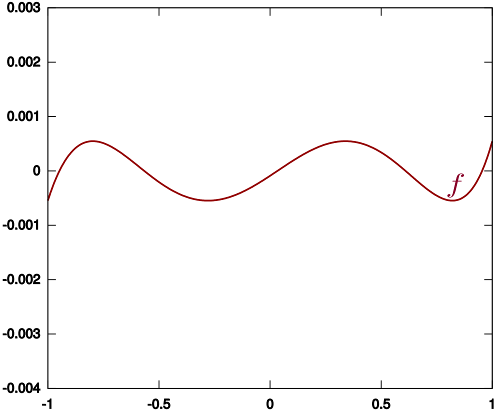


- Numerically compute best polynomial approximation p w.r.t $||\cdot||_\infty$.
- Certify *a posteriori* $||f - p||_\infty = \max_{[a,b]} |f(x) - p(x)|$.

Rigorous polynomial approximations (RPAs)

$||f - p|| \leq$



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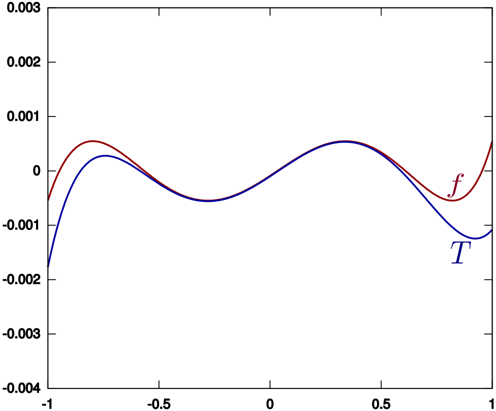
f replaced with
- polynomial approximation T (of higher degree, but easier to compute & certify)



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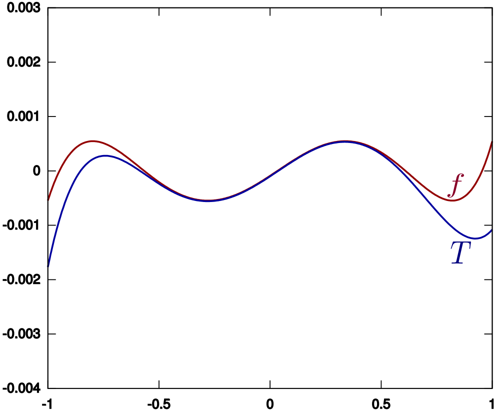

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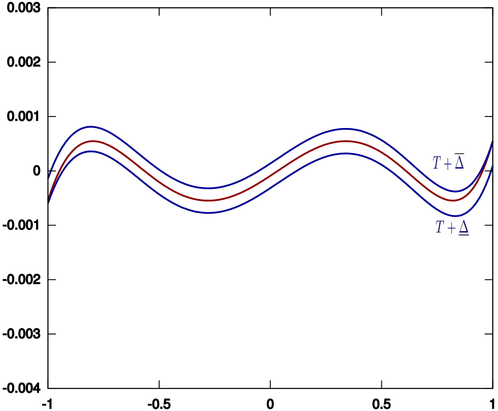

– Consider "sufficiently smooth" univariate functions f over $[a, b]$.
– f replaced with a rigorous polynomial approximation : $(T, \mathbf{\Delta})$

(1). RPAs based on Taylor series
⇝ Taylor Models (TMs).

(2). Near-best RPAs: based on Chebyshev Series
⇝ Chebyshev Models (CMs).

- f is an elementary function, e.g. $\exp(1/\cos(x))$;
- f is solution of a linear ordinary differential equation (with appropriate initial conditions).

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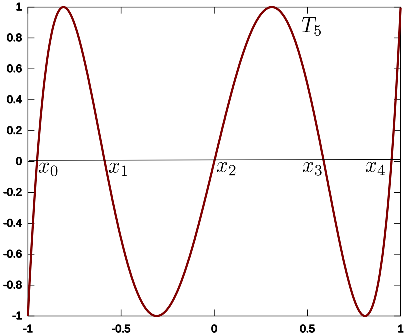
Chebyshev Polynomials

$$T_n(cos(\theta)) = \cos(n\theta)$$



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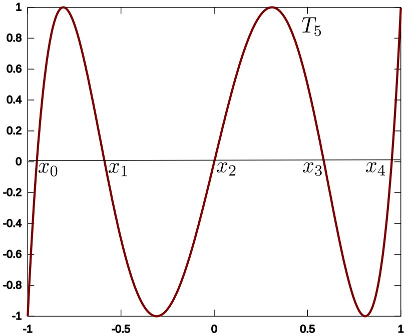

Chebyshev nodes: n distinct real roots in $[-1, 1]$ of T_n

$$x_k = \cos\left(\frac{(k+1/2)\,\pi}{n}\right), k = 0, \ldots, n-1.$$

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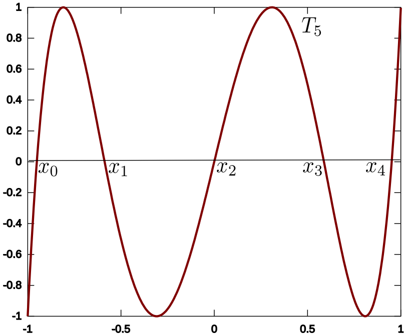

$T_{i+1} = 2xT_i - T_{i-1}, T_0(x) = 1, T_1(x) = x$

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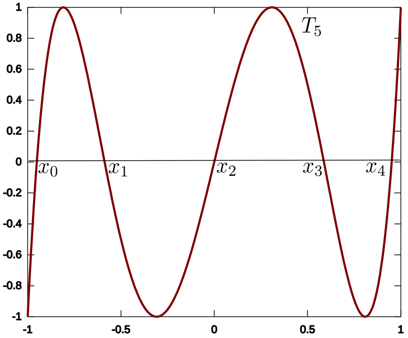

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Orthogonality:

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } i \neq j \\ \pi & \text{if } i = 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$$

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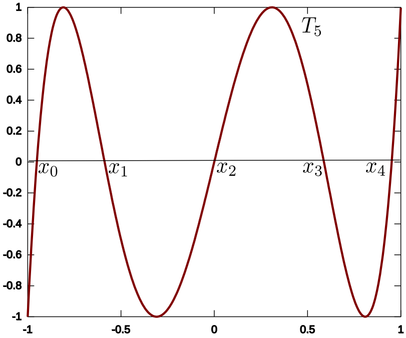

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Chebyshev Series vs Taylor Series

Two approximations of f:

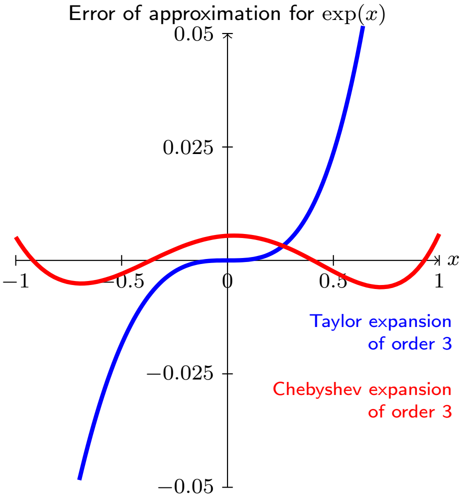
- by Taylor series

$$f = \sum_{n=0}^{+\infty} c_n x^n, \; c_n = \frac{f^{(n)}(0)}{n!},$$

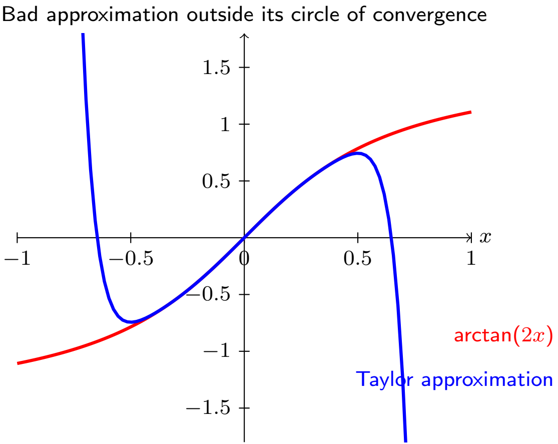
- or by Chebyshev series

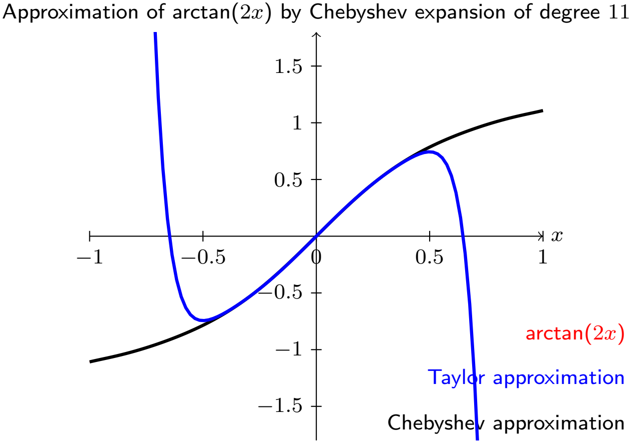
$$f = \sum_{n=-\infty}^{+\infty} t_n T_n(x),$$

$$t_n = \frac{1}{\pi} \int_{-1}^{1} T_n(t) \frac{f(t)}{\sqrt{1-t^2}} dt.$$



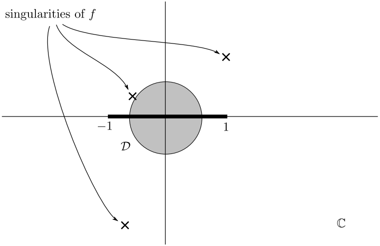
Chebyshev Series vs Taylor Series II



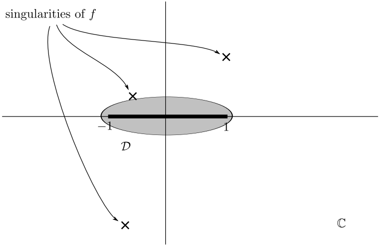

Approximation of arctan($2x$) by Chebyshev expansion of degree 11
1.5
1
0.5
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
x
arctan($2x$)
Taylor approximation
Chebyshev approximation

Convergence Domains :

For Taylor series:
disc centered at $x_0 = 0$ which avoids all the singularities of f



For Chebyshev series:
elliptic disc with foci at ± 1 which avoids all the singularities of f



Chebyshev Series vs Taylor Series III

Convergence Domains :

For Taylor series:
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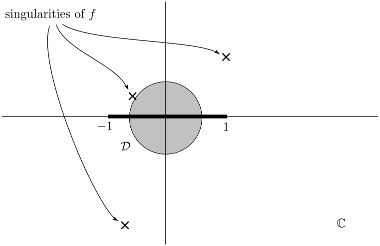

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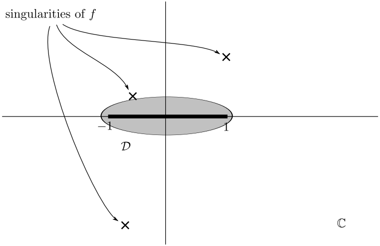

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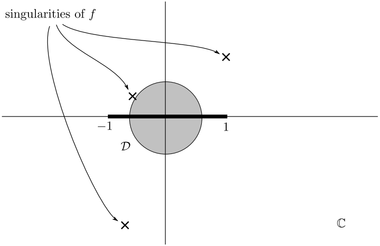

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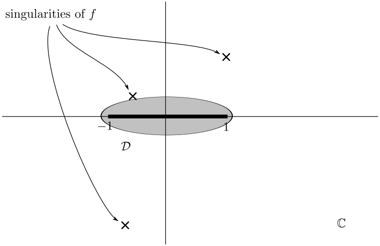

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✓ Chebyshev series converge over entire $[-1, 1]$ as soon as there are no real singularities in $[-1, 1]$.

Chebyshev Series vs Taylor Series IV

Truncation Error :

Taylor series, Lagrange formula:

$\forall x \in [-1,1],\ \exists \xi \in [-1,1]$ s.t.

$$f(x) - T(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

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[✓] We should have an improvement of 2^n in the width of the Chebyshev truncation error.

Quality of approximation of truncated Chebyshev series compared to best polynomial approximation

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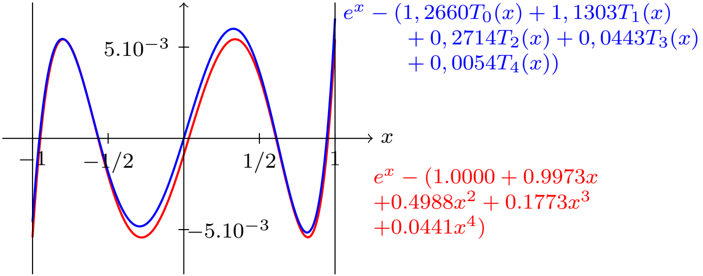
- $\Lambda_{10} = 2.22...$ $\to$ we lose at most 2 bits
- $\Lambda_{30} = 2.65...$ $\to$ we lose at most 2 bits
- $\Lambda_{100} = 3.13...$ $\to$ we lose at most 3 bits
- $\Lambda_{500} = 3.78...$ $\to$ we lose at most 3 bits



Chebyshev truncation of degree 4

Best approximant of degree 4

Chebyshev Series vs Taylor Series (9gag version)

Chebyshev series of $f = \sum_{i=-\infty}^{+\infty} t_i T_i(x)$:

TWO STEPS:

1. Obtain numerical approximation for coefficients of truncated Chebyshev series

   – Discrete orthogonality $\rightsquigarrow \widetilde{t_i} = \sum_{k=0}^{n} \frac{1}{n+1} f(x_k) T_i(x_k)$

   when f is elementary, evaluating f at Chebyshev nodes is easy

2. A posteriori validation of the solution with Banach Fixed Point Theorem (Newton-like Operator)

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   – When f is given by LODE: TODAY's Topic

2. A posteriori validation of the solution with Banach Fixed Point Theorem (Newton-like Operator)

- Infinite-dimensional linear problem:

$$\mathbf{L} = \partial^r + a_{r-1}\partial^{r-1} + \cdots + a_1\partial + a_0 \qquad : \mathcal{C}^r(I) \to \mathcal{C}^0(I),$$
$$\mathbf{B}_{t_0} : f \mapsto \Big(f(t_0), f'(t_0), \ldots, f^{(r-1)}(t_0)\Big) \qquad : \mathcal{C}^r(I) \to \mathbb{R}^r.$$

- Existence and Uniqueness of the Solution

Theorem 1 (Picard-Lindelöf – linear case)

The linear operator:

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is a (bicontinuous) isomorphism, which means that:

- *The solutions of the linear differential equation form a r-dimensional affine space.*
- *For fixed initial conditions at t_0, there is one and only one solution.*

Differentially-finite Functions (Stanley 1980)

Def.

A function $y : \mathbb{R} \to \mathbb{R}$ is D-finite if it is solution of a (homogeneous) linear differential equation with polynomial coefficients:

$$L \cdot y = a_r y^{(r)} + a_{r-1} y^{(r-1)} + \cdots + a_0 y = 0, \quad a_i \in \mathbb{R}[x].$$

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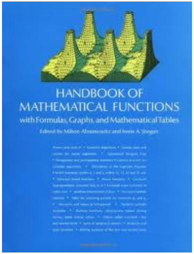

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The power of symbolic computation

Differential equation + initial conditions = Data Structure

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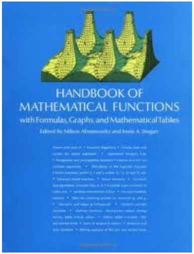
Fast algorithms for evaluation; Automatic proofs of identities



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A sequence (y_n) is P-recursive when it satisfies a recurrence relation of the form:

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Theorem

$\sum y_n x^n$ is solution of a linear differential equation with polynomial coefficients iff the sequence y_n is P-recursive.

Proof.

$$\begin{array}{rl} y(x) \leftrightarrow & y_n \\ \alpha y(x) & \alpha y_n \\ x y(x) & y_{n-1} \\ x y'(x) & n y_n \end{array}$$

e.g.

$$y' = y \leftrightarrow (n+1) y_{n+1} = y_n$$

D-finite functions

Closure

- Stable under operations: sum, product, Hadamard product, Laplace/Borel transform.
- y algebraic (exists a non-zero polynomial P s.t. $P(x, y) = 0$), f D-finite $\Rightarrow y$, $f \circ y$ D-finite

Some examples with gfun*

$\lambda \exp(x^k/k)$

$$y' = x^{k-1} y \leftrightarrow (n+1) y_{n+1} = y_{n-k+1}$$

diffeq order $= 1$; rec. order $= k \rightsquigarrow$ initial values determine the good subspace of solutions

* B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.

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diffeq order $=1$; rec. order $=k \rightsquigarrow$ initial values determine the good subspace of solutions

Compute the coefficient of x^{1000}:

$p(x) = (1+x)^{1000}(1+x+x^2)^{500}$

$$\frac{p'(x)}{p(x)} = \frac{1000}{1+x} + 500\frac{2x+1}{1+x+x^2}$$

* B. Salvy and P. Zimmermann. — Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. — ACM transactions on mathematical software, 1994.

D-finite functions: identity proof

$$\arcsin(x)^2 = \sum_{k\geq 0} \frac{k!}{\left(\frac{1}{2}\right)\dots\left(k+\frac{1}{2}\right)} \frac{x^{2k+2}}{2k+2}$$

- LODE for $\arcsin(x)$: $(1-x^2)y'' - xy' = 0, \quad y(0) = 0, \quad y'(0) = 1$

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- Let $h = y^2$:

$$\begin{aligned}
h' &= 2yy' \\
h'' &= 2y'^2 + 2yy'' = 2y'^2 + \frac{2x}{1-x^2}yy' \\
h''' &= 4y'y'' + \frac{2x}{1-x^2}(y'^2 + yy'') + \left(\frac{2}{1-x^2} + \frac{4x^2}{(1-x^2)^2}\right)yy' \\
&= \left(\frac{2}{1-x^2} + \frac{6x^2}{(1-x^2)^2}\right)yy' + \frac{6x}{1-x^2}y'^2
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- Vectors h, h', h'', h''' linear combination of 3 vectors y^2, yy', y'^2. Compute linear relation

$$(1-x^2)h''' - 3xh'' - h' = 0$$

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- Linear rec $(n+1)(n+2)(n+3)h_{n+3} - (n+1)^3 h_{n+1} = 0$
  easy to check (or to solve in this case)
  don't forget i.c. $h(0)=0, \quad h'(0)=0, \quad h''(0)=2.$

D-finite functions:

Many efficient symbolic algorithms:

- many special functions and combinatoric identities
- fast evaluation in arbitrary precision (analytic continuation)
- fast evaluation of P-recursive sequences.
- Software: algolib* (gfun, mgfun, numgfun) in Maple or HolonomicFunctions (C. Koutschan) in Mathematica
- Application: Web dictionary of special functions `http://ddmf.msr-inria.inria.fr`

Note: Examples in the previous slides thanks to B. Salvy's talks.

more

*`http://algo.inria.fr/libraries/`: B. Salvy, M. Mezzarobba, F. Chyzak, A. Bostan

Uniform Approximation of D-finite Functions

Problem

Given an integer d and a LODE (with polynomial coefficients) and suitable boundary conditions, find

the Chebyshev basis coefficients of a polynomial $p(x) = \sum_{0 \leq k \leq d} c_k T_k$ and a "small" bound B such that

$$|p(x) - f(x)| < B \text{ for all } x \text{ in } [-1, 1],$$

where f is the exact solution of the given LODE.

Related Works

Computation of the Chebyshev coefficients for D-finite functions

- Using a relation between coefficients Clenshaw (1957)
- Using the recurrence relation between the coefficients Fox-Parker (1968)
- The tau method of Lanczos (1938), Ortiz (1969-1993)

Today, *the computer algebra way* and *the numerical analyst way* (and their interaction)

Chebyshev Series of D-finite Functions

Theorem [60's, BenoitJoldesMezzarobba11]

$\sum u_n T_n(x)$ is solution of a linear differential equation with polynomial coefficients iff the sequence u_n is cancelled by a linear recurrence with polynomial coefficients.

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Recurrence relation + good initial conditions $\Rightarrow$ Fast numerical computation of the coefficients

Taylor: $\exp = \sum \frac{1}{n!} x^n$
Rec: $u(n+1) = \frac{u(n)}{n+1}$

$u(0) = 1$	$1/0! = 1$
$u(1) = 1$	$1/1! = 1$
$u(2) = 0,5$	$1/2! = 0,5$
$\vdots$	$\vdots$
$u(50) \approx 3,28.10^{-65}$	$1/50! \approx 3,28.10^{-65}$

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Chebyshev: $\exp = \sum I_n(1) T_n(x)$
Rec: $u(n+1) = -2nu(n) + u(n-1)$

$u(0) = 1,266$	$I_0(1) \approx 1,266$
$u(1) = 0,565$	$I_1(1) \approx 0,565$
$u(2) \approx 0,136$	$I_2(1) \approx 0,136$
$\vdots$	$\vdots$

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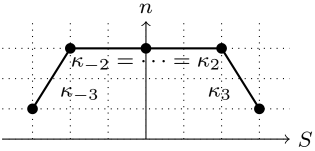
$u(2) \approx 0,136$	$I_2(1) \approx 0,136$
$\vdots$	$\vdots$
$u(50) \approx 4,450.10^{67}$	$I_{50}(1) \approx 2,934.10^{-80}$

Study of the Chebyshev recurrence

If $u(n)$ is solution, then there exists another solution $v(n) \sim \frac{1}{u(n)}$

Newton polygon of a Chebyshev recurrence



Convergent and Divergent Solutions of the Recurrence

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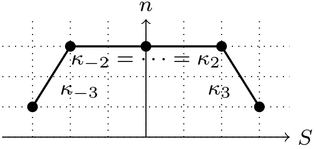

For the recurrence $u(n+1) + 2nu(n) - u(n-1)$

Two independent solutions are $I_n(1) \sim \frac{1}{(2n)!}$ and $K_n(1) \sim (2n)!$

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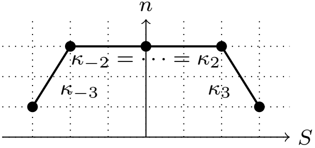

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Miller's algorithm

To compute the first N coefficients of the most convergent solution of a recurrence relation of order 2

- Initialize $u(N) = 0$ and $u(N-1) = 1$ and compute the first coefficients using the recurrence backwards
- Normalize u with the initial condition of the recurrence

Example 3

$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $\qquad c_{n+1} + 2n \, c_n - c_{n-1} = 0$

u_0	c_0
u_1	c_1
u_2	c_2
$\vdots$	$\vdots$
u_{50}	c_{50}
u_{51}	c_{51}
u_{52}	c_{52}

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$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n\, T_n(x)$ $\qquad c_{n+1} + 2n\, c_n - c_{n-1} = 0$

$$\begin{aligned} &u_0 \\ &u_1 \\ &u_2 \\ &\vdots \\ &u_{50} \\ &u_{51} = 1 \\ &u_{52} = 0 \end{aligned} \qquad\qquad \begin{aligned} &c_0 \\ &c_1 \\ &c_2 \\ &\vdots \\ &c_{50} \\ &c_{51} \\ &c_{52} \end{aligned}$$

Back to exp

Example 3

$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $c_{n+1} + 2n \, c_n - c_{n-1} = 0$

$$\begin{aligned} &u_0 \\ &u_1 \\ &u_2 \\ &\vdots \\ &u_{50} \approx 1,02 \cdot 10^2 \\ &u_{51} = 1 \\ &u_{52} = 0 \end{aligned}$$

$$\begin{aligned} &c_0 \\ &c_1 \\ &c_2 \\ &\vdots \\ &c_{50} \\ &c_{51} \\ &c_{52} \end{aligned}$$

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$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $\qquad c_{n+1} + 2n\, c_n - c_{n-1} = 0$

$$\begin{aligned}
u_0 & \\
u_1 & \\
u_2 & \approx -4,72 \cdot 10^{80} \\
& \vdots \\
u_{50} & \approx \quad 1,02 \cdot 10^{2} \\
u_{51} & = \quad 1 \\
u_{52} & = \quad 0
\end{aligned}
\qquad
\begin{aligned}
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& c_1 \\
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$$\begin{aligned}
u_0 & \\
u_1 &\approx 1,96 \cdot 10^{81} \\
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\qquad
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$\vdots$	$\vdots$
$u_{50} \approx 1,02 \cdot 10^2$	c_{50}
$u_{51} = 1$	c_{51}
$u_{52} = 0$	c_{52}

$$S = \sum_{n=-50}^{50} u_n\, T_n(0) \approx -3,48 \cdot 10^{81}$$

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$c_n := u_n / S$

$$\begin{aligned} c_0 &\approx 1,27 \\ c_1 &\approx -5,65 \cdot 10^{-1} \\ c_2 &\approx 1,36 \cdot 10^{-1} \\ &\vdots \\ c_{50} &\approx 2,93 \cdot 10^{-80} \\ c_{51} &\approx 2,88 \cdot 10^{-82} \\ c_{52} &\approx 0 \end{aligned}$$

$$S = \sum_{n=-50}^{50} u_n\, T_n(0) \approx -3,48 \cdot 10^{81}$$

Algorithm for Computing the Coefficients

Input: a differential equation of order r with boundary conditions
Output: a polynomial approximation of degree N of the solution

- compute the Chebyshev recurrence of order $2s \geq 2r$
- for i from 1 to s
  - using the recurrence relation backwards, compute the first N coefficients of the sequence $u^{[i]}$ starting with the initial conditions

$$\left(u^{[i]}(N+2s), \cdots, u^{[i]}(N+i), \cdots, u^{[i]}(N+1)\right) = (0, \cdots, 1, \cdots, 0)$$

- combine the s sequences $u^{[i]}$ according to the r boundary conditions and the $s-r$ symmetry relations

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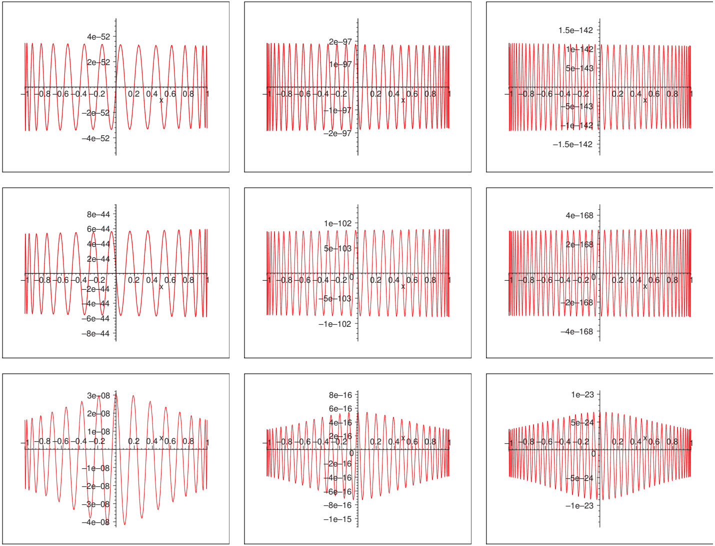
Theorem

This algorithm runs in $O(N)$ arithmetic operations



$\frac{e^{x/2}}{\sqrt{x+16}}$

$\frac{3\cos x - \sin x}{2}$

$e^{1/(1+2x^2)}$

degree = 30 | degree = 60 | degree = 90

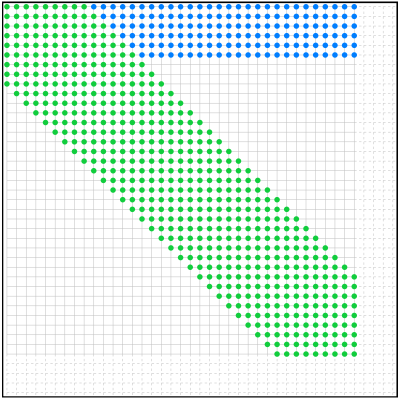
Computing the coefficients: The numerical analyst way

Boils down to efficiently solving a structured linear system*

$$a_r(x)y^{(r)}(x) + a_{r-1}(x)y^{(r-1)}(x) + \cdots + a_0(x)y(x) = 0 \text{ and initial conditions}$$

Equivalent to $\left(\mathbf{I} + \mathbf{K} \quad\right) \cdot \varphi = \psi$ where $\varphi = y^{(r)} := \sum_{k\geq 0} c_k T_k$

$$\mathbf{K} \cdot \sum_{k\geq 0} c_k T_k \simeq$$



$\cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_N \\ c_{N+1} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$

$\mathbf{K}$ is almost-banded and compact.

*Olver and Townsend Algorithm revisited

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Equivalent to $\left(\mathbf{I} + \mathbf{K}^{[\mathbf{N}]}\right) \cdot \varphi = \psi$ where $\varphi = y^{(r)} := \sum_{k\geq 0} c_k T_k$

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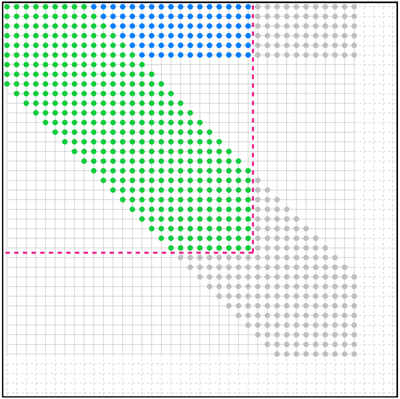


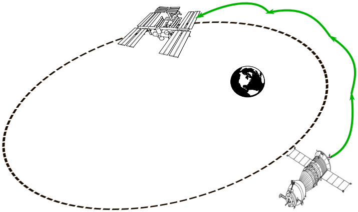
$$\cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ c_N \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

truncated integral operator $\mathbf{K}^{[N]}$.

*Olver and Townsend Algorithm revisited

Validated and numerically efficient Chebyshev Series Approximations for LODEs*

An example:



Linearized Equation of the In-Plane Motion

$$z''(t) + \left(4 - \frac{3}{1 + e\cos t}\right) z(t) = c$$

- Approximate solutions via Chebyshev series
- Validate approximate solutions with certified error bounds.

---

* Many thanks to D. Arzelier, F. Bréhard, N. Brisebarre.
Related articles BréhardBrisebarreJ18, ArantesBréhardGazzino18

Reformulation as an integral operator

- We want to solve $z''(t) + \left(4 - \frac{3}{1+0.5\cos t}\right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.

Reformulation as an integral operator

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  Equivalent to $\left(\mathbf{I} + \mathbf{K} \quad \right) \cdot \varphi = \psi$ where $\varphi = z''$.

Compact operator $\mathbf{K}$ in a suitable Banach space of Chebyshev coefficients

- $\mathbf{K} \cdot \varphi = t \left(4 - \dfrac{3}{1 + e\cos t}\right) \displaystyle\int_{-1}^{t} \varphi(s) \mathrm{d}s - \left(4 - \dfrac{3}{1 + e\cos t}\right) \displaystyle\int_{-1}^{t} s\varphi(s) \mathrm{d}s.$
- $\psi(t) = c - \big(z(-1) + (t+1)z'(-1)\big) \left(4 - \dfrac{3}{1 + e\cos t}\right).$

Reformulation as an integral operator

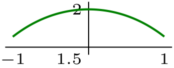
- We want to solve $z''(t) + \left(4 - \frac{3}{1+0.5\cos t}\right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.
- $\approx$ Equivalent to $\left(\mathbf{I} + \mathbf{K}^{[\mathbf{N}]}\right) \cdot \varphi = \psi$ where $\varphi = z''$.

Compact operator $\mathbf{K}$ in a suitable Banach space of Chebyshev coefficients

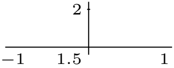
- $\mathbf{K} \cdot \varphi = t\left(4 - \dfrac{3}{1 + e\cos t}\right) \displaystyle\int_{-1}^{t} \varphi(s)\mathrm{d}s - \left(4 - \dfrac{3}{1 + e\cos t}\right) \int_{-1}^{t} s\varphi(s)\mathrm{d}s.$
- $\psi(t) = c - \left(z(-1) + (t+1)z'(-1)\right)\left(4 - \dfrac{3}{1 + e\cos t}\right).$

- How to get an efficient final dimensional approximation?

- Approximation of $t \mapsto 4 - \dfrac{3}{1 + e\cos t}$ over $[-1, 1]$ $(e = 0.5)$:



$\alpha(t)$



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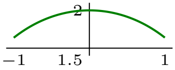
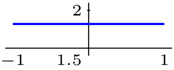


$\alpha(t)$

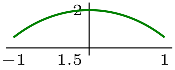


1.82

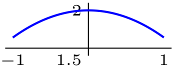
$$|\alpha(t) - 1.82| \leq 0.2$$

Approximating our Example

- Approximation of $t \mapsto 4 - \dfrac{3}{1 + e\cos t}$ over $[-1, 1]$ ($e = 0.5$):



$\alpha(t)$



$1.82 - 0.18T_2(t)$

$$|\alpha(t) - (1.82 - 0.18T_2(t))| \leq 0.007$$

The Almost-Banded Structure of the Operator $\mathbf{K}$ - Example

$$\mathbf{K} \cdot \varphi = t\left(4 - \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} \varphi(s)\mathrm{d}s + \left(-4 + \frac{3}{1 + e\cos t}\right)\int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

The Almost-Banded Structure of the Operator $\mathbf{K}$ - Example

$$\mathbf{K} \cdot \varphi \approx t(1.82 - 0.18T_2(t)) \int_{t_0}^{t} \varphi(s)\mathrm{d}s + (-1.82 + 0.18T_2(t)) \int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

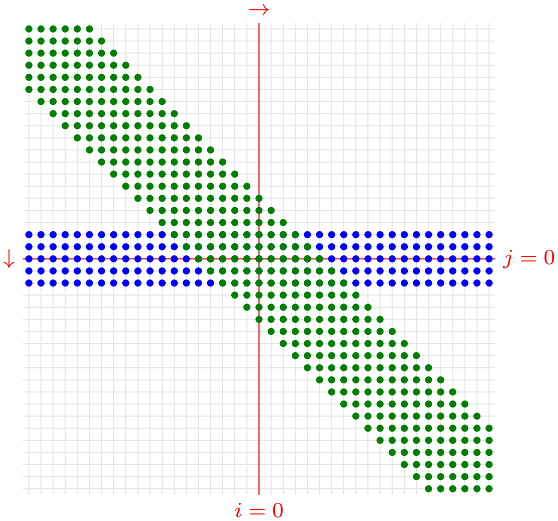
The Almost-Banded Structure of the Operator $\mathbf{K}$ - Example

$$\mathbf{K} \cdot \varphi \approx \underbrace{(1.73T_1(t) - 0.09T_3(t))}_{\beta_0(t)} \int_{t_0}^{t} \varphi(s)\mathrm{d}s + \underbrace{(-1.82 + 0.18T_2(t))}_{\beta_1(t)} \int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

$$\mathbf{K}\cdot\varphi \approx \underbrace{(1.73T_1(t) - 0.09T_3(t))}_{\beta_0(t)} \int_{t_0}^{t} \varphi(s)\mathrm{d}s + \underbrace{(-1.82 + 0.18T_2(t))}_{\beta_1(t)} \int_{t_0}^{t} s\varphi(s)\mathrm{d}s$$

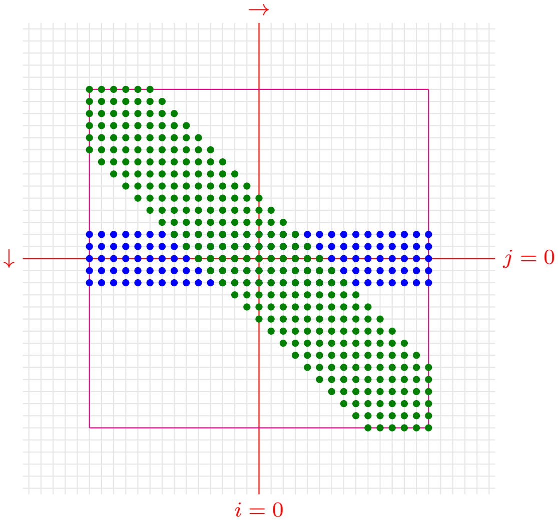
1.342500	-0.606667	-0.326250	0.364000	-0.155417	0.086667	-0.056875	0.040444	-0.030333	0.023636	-0.018958	0.015556	-0.013000	0.011030	-0.009479	0.008235	-0.007222	0.006386	-0.005687	0.005098	-0.004596
1.730000	-0.652500	-0.576667	0.438125	-0.115333	0.070208	-0.049429	0.036042	-0.027460	0.021625	-0.017475	0.014417	-0.012098	0.010298	-0.008872	0.007723	-0.006784	0.006007	-0.005356	0.004806	-0.004336
0.320000	0.060000	-0.271458	-0.036000	0.093833	-0.008571	0.004500	-0.004000	0.003000	-0.002338	0.001875	-0.001538	0.001286	-0.001091	0.000937	-0.000814	0.000714	-0.000632	0.000562	-0.000504	0.000455
-0.090000	0.109583	0.030000	-0.137375	0.006000	0.036042	0.002571	-0.002625	0.001429	-0.001125	0.000909	-0.000750	0.000629	-0.000536	0.000462	-0.000402	0.000353	-0.000313	0.000279	-0.000250	0.000226
-0.022500	0	0.052917	0	-0.065167	0	0.024036	0	-0.000536	0	0	0	0	0	0	0	0	0	0	0	0
0	-0.003750	0	0.028375	0	-0.040327	0	0.016104	0	-0.000402	0	0	0	0	0	0	0	0	0	0	0
0	0	-0.001875	0	0.018167	0	-0.027527	0	0.011548	0	-0.000312	0	0	0	0	0	0	0	0	0	0
0	0	0	-0.001125	0	0.012708	0	-0.020021	0	0.008688	0	-0.000250	0	0	0	0	0	0	0	0	0
0	0	0	0	-0.000750	0	0.009411	0	-0.015230	0	0.006774	0	-0.000205	0	0	0	0	0	0	0	0
0	0	0	0	0	-0.000536	0	0.007257	0	-0.011981	0	0.005431	0	-0.000170	0	0	0	0	0	0	0
0	0	0	0	0	0	-0.000402	0	0.005770	0	-0.009675	0	0.004451	0	-0.000144	0	0	0	0	0	0
0	0	0	0	0	0	0	-0.000312	0	0.004699	0	-0.007978	0	0.003715	0	-0.000124	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.000250	0	0.003902	0	-0.006692	0	0.003147	0	-0.000107	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.000205	0	0.003292	0	-0.005694	0	0.002701	0	-0.000094	0	0	0
0	0	0	0	0	0	0	0	0	0	-0.000170	0	0.002815	0	-0.004905	0	0.002343	0	-0.000083	0	0
0	0	0	0	0	0	0	0	0	0	0	-0.000144	0	0.002435	0	-0.004269	0	0.002052	0	-0.000074	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.000124	0	0.002127	0	-0.003749	0	0.001812	0	-0.000066
0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000107	0	0.001874	0	-0.003319	0	0.001612	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000094	0	0.001663	0	-0.002959	0	0.001443
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000083	0	0.001487	0	-0.002655	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000074	0	0.001337	0	-0.002395

The Almost-Banded Structure of the Operator $\mathbf{K}$



The infinite-dimensional operator $\mathbf{K}$.

The Almost-Banded Structure of the Operator $\mathbf{K}$



The final-dimensional truncation $\mathbf{K}^{[N]}$.

Reformulation as an integral operator

- We want to solve $z''(t) + \left(4 - \frac{3}{1+0.5\cos t}\right) z(t) = c$ with $z(-1) = 0$, $z'(-1) = 1$ and $c = 1$.

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  Equivalent to $\left(\mathbf{I} + \mathbf{K} \quad\right) \cdot \varphi = \psi$ where $\varphi = z''$.

Compact operator $\mathbf{K}$ in a suitable Banach space of Chebyshev coefficients

- $\mathbf{K} \cdot \varphi = t\left(4 - \frac{3}{1+e\cos t}\right)\int_{-1}^{t} \varphi(s)\mathrm{d}s - \left(4 - \frac{3}{1+e\cos t}\right)\int_{-1}^{t} s\varphi(s)\mathrm{d}s.$
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$$\begin{aligned}\widetilde{\varphi} = & -0.6T_0 - 1.19T_1 + 0.62T_2 + 0.17T_3 - 0.05T_4 - 0.01T_5 \\ & + 2.1\cdot 10^{-3}T_6 + 3.2\cdot 10^{-3}T_7 - 5.8\cdot 10^{-5}T_8 - 7.6\cdot 10^{-6}T_9 + 1.2\cdot 10^{-6}T_{10} \\ & + 1.4\cdot 10^{-7}T_{11} - 1.9\cdot 10^{-8}T_{12} - 2.0\cdot 10^{-9}T_{13} + 2.6\cdot 10^{-10}T_{14} + 2.5\cdot 10^{-11}T_{15} \\ & - 3.0\cdot 10^{-12}T_{16} - 2.6\cdot 10^{-13}T_{17} + 3.0\cdot 10^{-14}T_{18} + 2.5\cdot 10^{-15}T_{19} - 2.6\cdot 10^{-16}T_{20}\end{aligned}$$

- Recall: For the integral equation of unknown φ

$$(\mathbf{I} + \mathbf{K}) \cdot \varphi = \psi,$$

we want to validate an approximate solution $\widetilde{\varphi}$, in a suitable Banach space Ч^1:

$$\|\widetilde{\varphi} - \varphi^*\|_{\text{Ч}^1}.$$

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- Reformulation as a fixed point equation:

$$\varphi+\mathbf{K}\cdot\varphi=\psi \Leftrightarrow \mathbf{T}\cdot\varphi=\varphi,$$

$$\mathbf{T}\cdot\varphi=\varphi-\mathbf{A}\cdot(\varphi+\mathbf{K}\cdot\varphi-\psi)\,, \qquad \mathbf{A}\approx(\mathbf{I}+\mathbf{K})^{-1} \text{ injective.}$$

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- If $\|\mathrm{D}\mathbf{T}\|_{Ч^1}=\|\mathbf{I}-\mathbf{A}\,(\mathbf{I}+\mathbf{K})\|_{Ч^1}=k<1$, $\mathbf{T}$ is contractive and we get a tight enclosure of the approximation error:

$$\frac{\|\mathbf{T}\cdot\widetilde{\varphi}-\widetilde{\varphi}\|_{Ч^1}}{1+k}\leq\|\widetilde{\varphi}-\varphi^*\|_{Ч^1}\leq\frac{\|\mathbf{T}\cdot\widetilde{\varphi}-\widetilde{\varphi}\|_{Ч^1}}{1-k}.$$

We are looking for an approximate inverse matrix:

$$\mathbf{A} \approx (\mathbf{I} + \mathbf{K})^{-1}.$$

0.161442	0.292063	0.309506	-0.202732	0.032680	-0.025359	0.022468	-0.016198	0.012178	-0.009592	0.007748	-0.006386	0.005355	-0.004555	0.003922	-0.003413	0.002997	-0.002652	0.002364	-0.002123	0.001915
-0.951737	1.182179	0.521477	-0.177420	-0.063659	0.010819	0.004034	-0.002709	0.002226	-0.002039	0.001797	-0.001567	0.001369	-0.001201	0.001059	-0.000939	0.000836	-0.000749	0.000675	-0.000611	0.000555
0.013779	-0.233695	1.197801	0.162217	-0.147560	0.020291	-0.013304	0.012961	-0.009866	0.007675	-0.006197	0.005110	-0.004285	0.003645	-0.003138	0.002731	-0.002398	0.002122	-0.001892	0.001699	-0.001532
0.137573	-0.112050	-0.075379	1.156558	0.009202	-0.048363	-0.000583	0.002026	-0.000322	0.000261	-0.000260	0.000227	-0.000198	0.000174	-0.000153	0.000136	-0.000121	0.000108	-0.000098	0.000088	-0.000080
0.003107	0.020279	-0.060441	-0.014077	1.079375	-0.001761	-0.025165	-0.001125	0.001745	-0.000666	0.000523	-0.000443	0.000372	-0.000316	0.000272	-0.000237	0.000208	-0.000184	0.000164	-0.000147	0.000133
-0.007791	0.007936	0.004269	-0.034920	-0.000521	1.043723	0.000033	-0.017202	0.000018	0.000559	0.000015	-0.000020	0.000011	-0.000010	0.000009	-0.000008	0.000007	-0.000006	0.000006	-0.000005	0.000005
-0.000031	-0.000830	0.003439	0.000576	-0.020460	0.000072	1.028868	0.000046	-0.012114	0.000027	0.000385	0.000018	-0.000020	0.000013	-0.000011	0.000010	-0.000009	0.000008	-0.000007	0.000006	-0.000005
0.000259	-0.000232	-0.000142	0.001781	0.000017	-0.013596	-0.000001	1.020722	-0.000001	-0.008981	-0.000000	0.000307	-0.000000	-0.000002	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000	-0.000000
0.000003	0.000023	-0.000079	-0.000016	0.001018	-0.000002	-0.009855	-0.000001	1.015624	-0.000001	-0.006950	-0.000001	0.000241	-0.000000	-0.000001	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000
-0.000006	0.000006	0.000003	-0.000032	-0.000000	0.000666	0.000000	-0.007508	0.000000	1.012219	0.000000	-0.005544	0.000000	0.000194	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000002	0.000000	-0.000014	0.000000	0.000475	0.000000	-0.005923	0.000000	1.009828	0.000000	-0.004527	0.000000	0.000161	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000001	0.000000	-0.000007	-0.000000	0.000357	-0.000000	-0.004798	-0.000000	1.008081	-0.000000	-0.003767	-0.000000	0.000135	-0.000000	-0.000001	-0.000000	0.000000	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000004	-0.000000	0.000279	-0.000000	-0.003969	-0.000000	1.006764	-0.000000	-0.003185	-0.000000	0.000116	-0.000000	-0.000000	-0.000000	0.000000
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000003	0.000000	0.000224	0.000000	-0.003339	0.000000	1.005746	0.000000	-0.002729	0.000000	0.000100	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000002	0.000000	0.000184	0.000000	-0.002849	0.000000	1.004943	0.000000	-0.002364	0.000000	0.000088	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000154	-0.000000	-0.002460	-0.000000	1.004298	-0.000000	-0.002068	-0.000000	0.000077	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000131	-0.000000	-0.002146	-0.000000	1.003772	-0.000000	-0.001825	-0.000000	0.000069
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000001	0.000000	0.000113	0.000000	-0.001888	0.000000	1.003337	0.000000	-0.001622	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000098	0.000000	-0.001675	0.000000	1.002973	0.000000	-0.001451
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000086	-0.000000	-0.001496	-0.000000	1.002664	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000076	-0.000000	-0.001344	-0.000000	1.002403

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$$\mathbf{A} \approx (\mathbf{I} + \mathbf{K}^{[N]})^{-1}.$$

0.161442	0.292063	0.309506	-0.202732	0.032680	-0.025359	0.022468	-0.016198	0.012178	-0.009592	0.007748	-0.006386	0.005355	-0.004555	0.003922	-0.003413	0.002997	-0.002652	0.002364	-0.002123	0.001915
-0.951737	1.182179	0.521477	-0.177420	-0.063659	0.010819	0.004034	-0.002709	0.002226	-0.002039	0.001797	-0.001567	0.001369	-0.001201	0.001059	-0.000939	0.000836	-0.000749	0.000675	-0.000611	0.000555
0.013779	-0.233695	1.197801	0.162217	-0.147560	0.020291	-0.013304	0.012961	-0.009866	0.007675	-0.006197	0.005110	-0.004285	0.003645	-0.003138	0.002731	-0.002398	0.002122	-0.001892	0.001699	-0.001532
0.137573	-0.112050	-0.075379	1.156558	0.009202	-0.048363	-0.000583	0.002026	-0.000322	0.000261	-0.000260	0.000227	-0.000198	0.000174	-0.000153	0.000136	-0.000121	0.000108	-0.000098	0.000088	-0.000080
0.003107	0.020279	-0.060441	-0.014077	1.079375	-0.001761	-0.025165	-0.001125	0.001745	-0.000666	0.000523	-0.000443	0.000372	-0.000316	0.000272	-0.000237	0.000208	-0.000184	0.000164	-0.000147	0.000133
-0.007791	0.007936	0.004269	-0.034920	-0.000521	1.043723	0.000033	-0.017202	0.000018	0.000559	0.000015	-0.000020	0.000011	-0.000010	0.000009	-0.000008	0.000007	-0.000006	0.000006	-0.000005	0.000005
-0.000031	-0.000830	0.003439	0.000576	-0.020460	0.000072	1.028868	0.000046	-0.012114	0.000027	0.000385	0.000018	-0.000020	0.000013	-0.000011	0.000010	-0.000009	0.000008	-0.000007	0.000006	-0.000005
0.000259	-0.000232	-0.000142	0.001781	0.000017	-0.013596	-0.000001	1.020722	-0.000001	-0.008981	-0.000000	0.000307	-0.000000	-0.000002	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000	-0.000000
0.000003	0.000023	-0.000079	-0.000016	0.001018	-0.000002	-0.009855	-0.000001	1.015624	-0.000001	-0.006950	-0.000001	0.000241	-0.000000	-0.000001	-0.000000	0.000000	-0.000000	0.000000	-0.000000	0.000000
-0.000006	0.000006	0.000003	-0.000032	-0.000000	0.000666	0.000000	-0.007508	0.000000	1.012219	0.000000	-0.005544	0.000000	0.000194	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000002	0.000000	-0.000014	0.000000	0.000475	0.000000	-0.005923	0.000000	1.009828	0.000000	-0.004527	0.000000	0.000161	0.000000	-0.000001	0.000000	0.000000	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000001	0.000000	-0.000007	-0.000000	0.000357	-0.000000	-0.004798	-0.000000	1.008081	-0.000000	-0.003767	-0.000000	0.000135	-0.000000	-0.000001	-0.000000	0.000000	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000004	-0.000000	0.000279	-0.000000	-0.003969	-0.000000	1.006764	-0.000000	-0.003185	-0.000000	0.000116	-0.000000	-0.000000	-0.000000	0.000000
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000003	0.000000	0.000224	0.000000	-0.003339	0.000000	1.005746	0.000000	-0.002729	0.000000	0.000100	0.000000	-0.000000	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000002	0.000000	0.000184	0.000000	-0.002849	0.000000	1.004943	0.000000	-0.002364	0.000000	0.000088	0.000000	-0.000000
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000154	-0.000000	-0.002460	-0.000000	1.004298	-0.000000	-0.002068	-0.000000	0.000077	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000001	-0.000000	0.000131	-0.000000	-0.002146	-0.000000	1.003772	-0.000000	-0.001825	-0.000000	0.000069
-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000001	0.000000	0.000113	0.000000	-0.001888	0.000000	1.003337	0.000000	-0.001622	0.000000
-0.000000	-0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000000	0.000000	-0.000000	0.000000	0.000098	0.000000	-0.001675	0.000000	1.002973	0.000000	-0.001451
0.000000	-0.000000	-0.000000	0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000086	-0.000000	-0.001496	-0.000000	1.002664	-0.000000
0.000000	0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000000	-0.000000	-0.000000	-0.000000	0.000076	-0.000000	-0.001344	-0.000000	1.002403

We are looking for an approximate inverse matrix:

$$\mathbf{A} \approx (\mathbf{I} + \mathbf{K}^{[N]})^{-1}.$$

0.161442	0.292063	0.309506	-0.202732	0.032680	-0.025359	0.022468	-0.016198	0.012178	-0.009592	0.007748	-0.006386	0.005355	-0.004555	0.003922	-0.003413	0.002997	-0.002652	0.002364	-0.002123	0.001915
-0.951737	1.182179	0.521477	-0.177420	-0.063659	0.010819	0.004034	-0.002709	0.002226	-0.002039	0.001797	-0.001567	0.001369	-0.001201	0.001059	-0.000939	0.000836	-0.000749	0.000675	-0.000611	0.000555
0.013779	-0.233695	1.197801	0.162217	-0.147560	0.020291	-0.013304	0.012961	-0.009866	0.007675	-0.006197	0.005110	-0.004285	0.003645	-0.003138	0.002731	-0.002398	0.002122	-0.001892	0.001699	-0.001532
0.137573	-0.112050	-0.075379	1.156558	0.009202	-0.048363	-0.000583	0.002026	0	0	0	0	0	0	0	0	0	0	0	0	0
0.003107	0.020279	-0.060441	-0.014077	1.079375	-0.001761	-0.025165	-0.001125	0.001745	-0.000666	0.000523	0	0	0	0	0	0	0	0	0	0
-0.007791	0.007936	0.004269	-0.034920	0	1.043723	0	-0.017202	0	0.000559	0	0	0	0	0	0	0	0	0	0	0
0	0	0.003439	0	-0.020460	0	1.028868	0	-0.012114	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.001781	0	-0.013596	0	1.020722	0	-0.008981	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.001018	0	-0.009855	0	1.015624	0	-0.006950	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.000666	0	-0.007508	0	1.012219	0	-0.005544	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-0.005923	0	1.009828	0	-0.004527	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-0.004798	0	1.008081	0	-0.003767	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-0.003969	0	1.006764	0	-0.003185	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-0.003339	0	1.005746	0	-0.002729	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-0.002849	0	1.004943	0	-0.002364	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	-0.002460	0	1.004298	0	-0.002068	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.002146	0	1.003772	0	-0.001825	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001888	0	1.003337	0	-0.001622	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001675	0	1.002973	0	-0.001451
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001496	0	1.002664	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.001344	0	1.002403

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\mathrm{Y}^1} \le \underbrace{\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\mathrm{Y}^1}}_{\text{Approximation error}} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\mathrm{Y}^1}.$$

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\Upsilon^1} \leq \underbrace{\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\Upsilon^1}}_{\text{Approximation error}} + \underbrace{\|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\Upsilon^1}}_{\text{Truncation error}}.$$

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\mathrm{Y}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\mathrm{Y}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\mathrm{Y}^1}.$$

- Addition, Multiplication and 1-norm of almost-banded matrices: linear in N.

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\mathrm{Y}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\mathrm{Y}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\mathrm{Y}^1}.$$

- Addition, Multiplication and 1-norm of almost-banded matrices: linear in N.
- Hence, this certification step is linear in N.

Computing the Operator Norm (1/2)

- Decomposition of the operator norm:

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

- Addition, Multiplication and 1-norm of almost-banded matrices: linear in N.
- Hence, this certification step is linear in N.

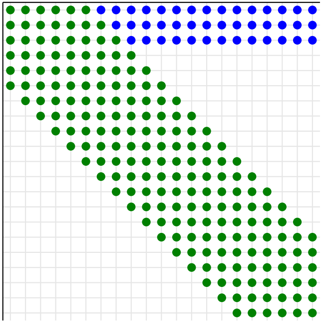
Example

In our case, the approximation error is:

$$\|\mathbf{I} - \mathbf{A}\left(\mathbf{I} + \mathbf{K}^{[N]}\right)\|_{\text{Ч}^1} \leq \mathbf{1.5 \cdot 10^{-3}}$$

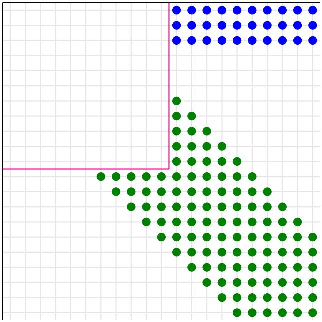
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{Ч}^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{Ч}^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{Ч}^1}.$$

$$\|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\text{ч}1} \leq \|\mathbf{I}-\mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\text{ч}1} + \|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\text{ч}1}.$$



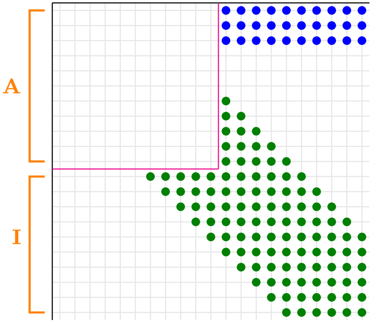
$\mathbf{K}$

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



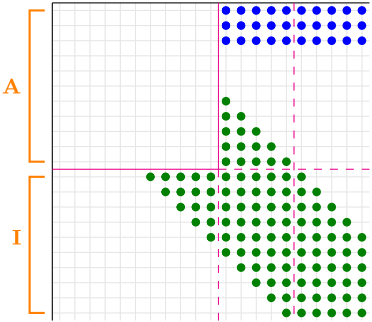
$\mathbf{K} - \mathbf{K}^{[N]}$

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\text{ч}1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\text{ч}1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\text{ч}1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

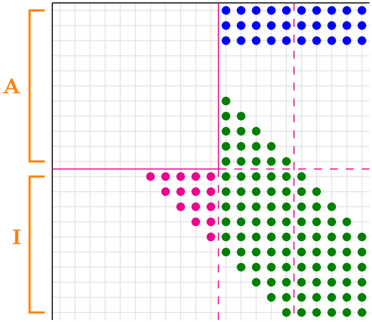
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

Computing the Operator Norm (2/2)

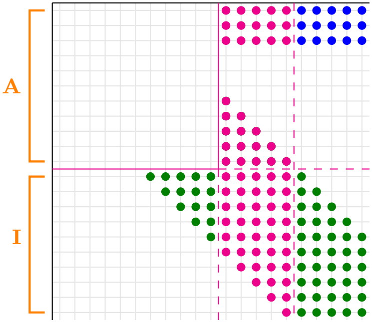
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\mho^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\mho^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\mho^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.

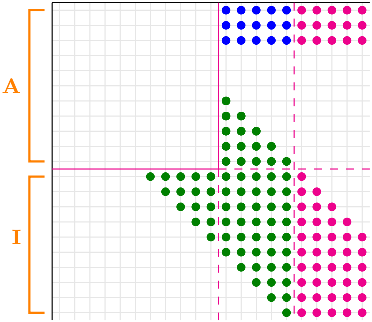
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K})\|_{\ell^1} \le \|\mathbf{I} - \mathbf{A}(\mathbf{I}+\mathbf{K}^{[N]})\|_{\ell^1} + {\color{red}\|\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})\|_{\ell^1}}.$$



$\mathbf{A}(\mathbf{K}-\mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$

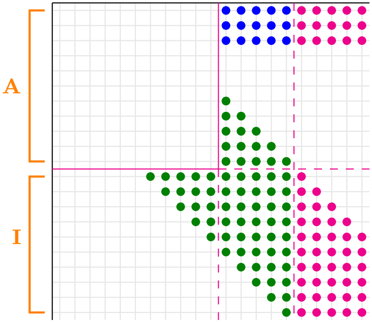


$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Computing the Operator Norm (2/2)

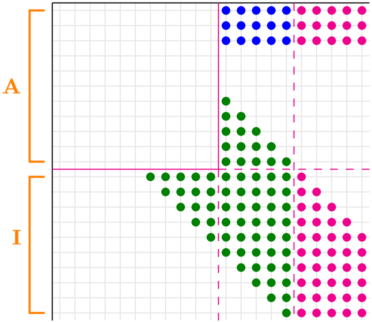
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

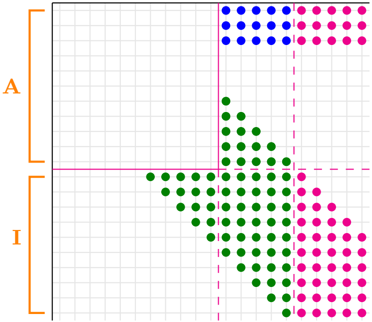
$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$

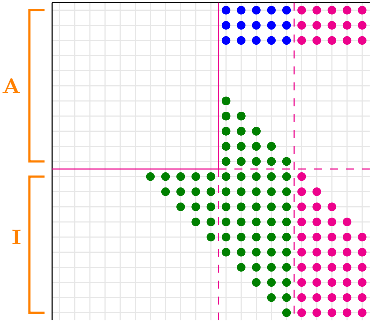


$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Truncation error of the example

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



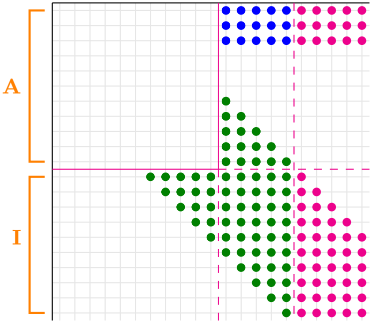
$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Truncation error of the example

$1.3 \cdot 10^{-3}$

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

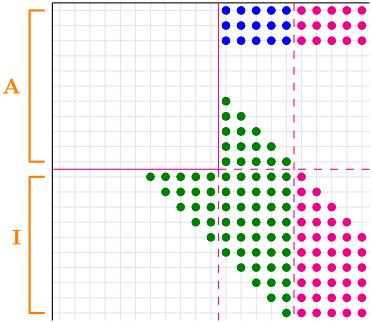
- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Truncation error of the example

$1.3 \cdot 10^{-3}$ $5.2 \cdot 10^{-3}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

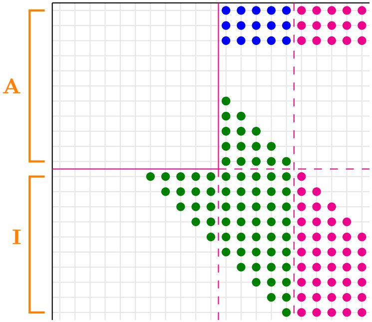
- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Truncation error of the example

$1.3 \cdot 10^{-3}$ $5.2 \cdot 10^{-3}$ $9.4 \cdot 10^{-3} + 2.7 \cdot 10^{-3}$

Computing the Operator Norm (2/2)

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + \|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}.$$



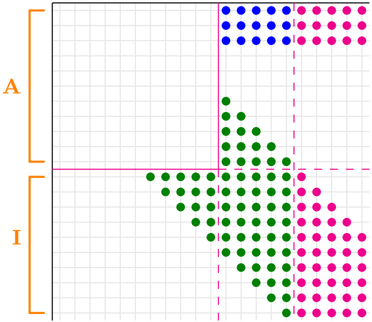
$\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})$

- Direct computation.
- Apply $\mathbf{A}$ and direct computation.
- Bound the remaining *infinite* number of colums based on decrease like $1/i$ and $1/i^2$

Truncation error of the example

$1.3 \cdot 10^{-3}$ $\quad$ $5.2 \cdot 10^{-3}$ $\quad$ $1.21 \cdot 10^{-2}$

$$\|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K})\|_{\ell^1} \leq \|\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{K}^{[N]})\|_{\ell^1} + {\color{red}\|\mathbf{A}(\mathbf{K} - \mathbf{K}^{[N]})\|_{\ell^1}}.$$



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$1.3 \cdot 10^{-3}$ $\quad 5.2 \cdot 10^{-3}$ $\quad 1.21 \cdot 10^{-2}$ $\quad \Rightarrow \quad$ $\mathbf{1.21 \cdot 10^{-2}}$

Bring our Example to the End

- $\boldsymbol{k} \leq \mathbf{1.5 \cdot 10^{-3}} + \mathbf{1.21 \cdot 10^{-2}}$.

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- Hence:

$$\frac{6.48 \cdot 10^{-16}}{1 + \boldsymbol{k}} \leq \|\widetilde{\varphi} - \varphi^*\|_{\mho^1} \leq \frac{6.48 \cdot 10^{-16}}{1 - \boldsymbol{k}}$$

Bring our Example to the End

- $\boldsymbol{k \le 1.36 \cdot 10^{-2}}$.
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- Hence:

$$6.39 \cdot 10^{-16} \quad \le \quad \|\widetilde{\varphi} - \varphi^*\|_{\text{Ч}^1} \quad \le \quad 6.57 \cdot 10^{-16}$$

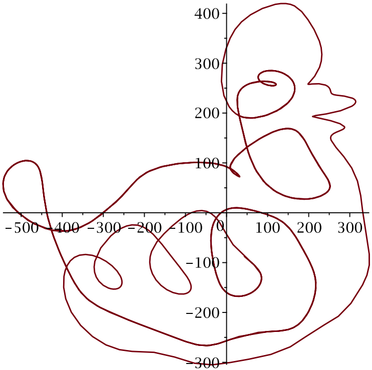
Conclusion

- Ask F. Bréhard for TchebyApprox: a C library to compute certified approximations to solutions of Linear Ordinary Differential Equations, using truncated Chebyshev series & Generalisation to Systems of LODEs
  https://gforge.inria.fr/projects/tchebyapprox

- D-finite functions provide a rich framework and structure

- P-recursive sequences for other orthogonal basis expansions

Even ducks are D-finite!



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Examples of recurrences* †

```
> with(gfun):
> deq:=holexprtodiffeq(sin(x), y(x));
```

$$deq := \left\{ \frac{d^2}{dx^2} y(x) + y(x), y(0) = 0, D(y)(0) = 1 \right\}$$

```
> diffeqtorec(deq, y(x), u(k));
```

$$\left\{ u(k) + \left(k^2 + 3k + 2\right) u(k+2), u(0) = 0, u(1) = 1 \right\}$$

* B. Salvy and P. Zimmermann. – Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. ACM transactions on mathematical software, 1994.

† A. Benoit and B. Salvy. – Chebyshev expansions for solutions of linear differential equations. In J. P. May, editor, ISSAC '09, pages 23-30. ACM, 2009.

Examples of recurrences* †

```
> deq:=holexprtodiffeq((x^2+2*x+4)*sin(x)+exp(x)/(x+2), y(x));
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$$deq := \left\{ \left(-x^7 - 6x^6 - 32x^5 - 112x^4 - 228x^3 - 236x^2 - 96x\right) y(x) + \left(x^7 + 11x^6 + 58x^5 + 206x^4 + 452x^3 + \right.\right.$$

$$- 320x^3 - 440x^2 - 352x - 112) \left(\frac{d^2}{dx^2} y(x)\right) + \left(x^7 + 7x^6 + 28x^5 + 84x^4 + 176x^3 + 232x^2 + 176x + 64\right.$$

```
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$$\left\{ -u(k) + (k-5)\,u(k+1) + \left(-k^2 + 8k - 12\right) u(k+2) + \left(k^3 - 4k^2 + 19k + 8\right) u(k+3) + \left(7k^3 + 13k^2\right.\right.$$

$$+ \left(84k^3 + 940k^2 + 3272k + 3840\right) u(k+6) + \left(176k^3 + 2728k^2 + 13480k + 21056\right) u(k+7) + \left(232k^3\right.$$

$$+ 80640) u(k+9) + \left(64k^3 + 1728k^2 + 15488k + 46080\right) u(k+10), u(0) = \frac{1}{2}, u(1) = \frac{17}{4}, u(2) = \frac{17}{8}, u($$

$$-\frac{1}{7680}, u(9) = -\frac{103}{322560} \left.\right\}$$

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```
> gfsRecurrence[diffeqToGFSRec](deq, y(x), u(n), functions=ChebyshevT(n,x));
```

$$(-n^2 - 23n - 132)\, u(n) + (2n^3 + 36n^2 + 34n - 1320)\, u(n+1) + (-4n^4 - 60n^3 + 156n^2 + 3022n - 6936)\, u(n+2) + (8n^5 + 152n^4 + 482n^3 - 428n^2 + 22826n + 72$$

$$+ 3) + (112n^5 + 2760n^4 + 19616n^3 + 41261n^2 + 148189n + 370944)\, u(n+4) + (952n^5 + 28056n^4 + 279928n^3 + 1130928n^2 + 2254672n + 3036552)\, u(n+5) + (60$$

$$+ 206088n^4 + 2561712n^3 + 14269656n^2 + 35990280n + 35179608)\, u(n+6) + (27176n^5 + 1042584n^4 + 15178936n^3 + 104070256n^2 + 334823360n + 404112360)\, u(n+$$

$$+ (82576n^5 + 3498216n^4 + 57475040n^3 + 456412158n^2 + 1748129922n + 2579039376)\, u(n+8) + (166936n^5 + 7722200n^4 + 140212972n^3 + 1247689432n^2 + 543815587$$

$$+ 9287865144)\, u(n+9) + (218816n^5 + 10940800n^4 + 215814744n^3 + 2098122320n^2 + 10051489972n + 18991011720)\, u(n+10) + (166936n^5 + 8971400n^4 + 190180972$$

$$+ 1986688888n^2 + 10224544996n + 20746455576)\, u(n+11) + (82576n^5 + 4759384n^4 + 107921760n^3 + 1202451842n^2 + 6579579602n + 14147655864)\, u(n+12) + (271$$

$$+ 1675016n^4 + 40476216n^3 + 478544304n^2 + 2764848320n + 6245500440)\, u(n+13) + (6048n^5 + 398712n^4 + 10266672n^3 + 128661864n^2 + 782842440n + 1849979592)$$

$$+ 14) + (952n^5 + 67144n^4 + 1843448n^3 + 24490352n^2 + 156739152n + 386549688)\, u(n+15) + (112n^5 + 8440n^4 + 246816n^3 + 3471699n^2 + 23316949n + 59816436)$$

$$+ 16) + (8n^5 + 648n^4 + 20322n^3 + 304548n^2 + 2154346n + 5756496)\, u(n+17) + (4n^4 + 260n^3 + 5844n^2 + 52782n + 164976)\, u(18+n) + (2n^3 + 84n^2 + 994n$$

$$+ 3600)\, u(19+n) + (n^2 + 17n + 72)\, u(n+20)$$

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* **B. Salvy and P. Zimmermann. – Gfun: a Maple package for the manipulation of generating and holonomic functions in**

D-finite functions

Prove that $\sec = \frac{1}{\cos}$ is not D-finite

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- $\rightsquigarrow$ non-zero polynomial equation in x, y, and $\sqrt{y^2 - 1}$ satisfied by $y \rightsquigarrow$ y algebraic $\rightsquigarrow$ contradiction