Global instability and scattering maps

Winter Workshop on Dynamics, Topology and Computations 2018

Amadeu Delshams

Departament de Matemàtiques and Lab of Geometry and Dynamical Systems
Universitat Politècnica de Catalunya

Jan. 29-Feb. 2, 2018

In 1964, V.I. Arnold proposed an example of a nearly-integrable Hamiltonian with $2+1/2$ degreees of freedom

$$H(q,p,\varphi,I,t) = \frac{1}{2}\left(p^2+I^2\right) + \varepsilon(\cos q - 1)\left(1+\mu(\sin\varphi + \cos t)\right),$$

and asserted [Arnold64] that given any $\delta, K > 0$, for any $0 < \mu \ll \varepsilon \ll 0$, there exists a trajectory of this Hamiltonian system such that

$$I(0) < \delta \text{ and } I(T) > K \qquad \text{for some time } T > 0.$$

Notice that this a global instability result for the variable I, since

$$\dot{I} = -\frac{\partial H}{\partial \varphi} = -\varepsilon\mu(\cos q - 1)\cos\varphi$$

is zero for $\varepsilon = 0$, so I remains constant, whereas I can have a drift of finite size for any $\varepsilon > 0$ small enough.

Arnold's Hamiltonian can be written as a nearly-integrable with 3 degrees of freedom

$$H^*(q,p,\varphi,I,s,A) = \frac{1}{2}\left(p^2+I^2\right) + A + \varepsilon(\cos q - 1)\left(1+\mu(\sin\varphi + \cos s)\right),$$

which for $\varepsilon = 0$ is an integrable Hamiltonian $h(p,I,A) = \frac{1}{2}\left(p^2+I^2\right)+A$. Since h satisfies the (Arnold) isoenergetic nondegeneracy

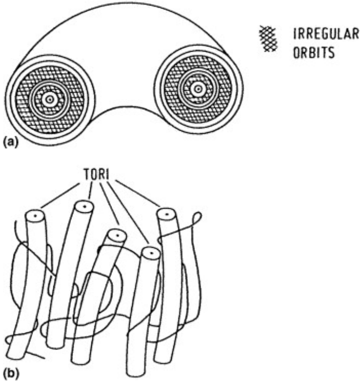
$$\begin{vmatrix} D^2h & Dh \\ Dh^\top & 0 \end{vmatrix} = -1 \neq 0$$

By the KAM theorem [Arnold63] proven by Arnold in 1963, any 5D energy level $H =$ const. is filled, up to a set of relative measure $\mathrm{O}(\sqrt{\varepsilon})$, with 3D-invariant tori $\mathcal{T}_\omega$ with Diophantine frequencies $\omega = (\omega_1,\omega_2,1)$:

$$|k_1\omega_1 + k_2\omega_2 + k_0| \geq \gamma/|k|^\tau \text{ for any } 0 \neq (k_1,k_2,k_0) \in \mathbb{Z},$$

where $\gamma = \mathrm{O}(\sqrt{\varepsilon})$, and $\tau \geq 2$.

Figure: a): 2D tori separate a 3D phase space. b) 3D tori do not separate a 5D phase space



Since the 3D KAM invariant tori do not separate the 5D phase space, there can exist irregular orbits 'traveling' between tori. Arnold conjectured in the KAM theorem in 1963 that this was the general case.

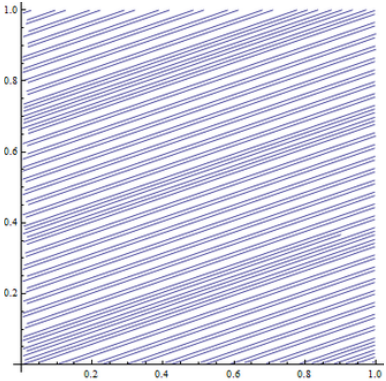
The unperturbed rôle is played by a (completely) integrable Hamiltonian with n degrees of freedom. The Liouville–Arnold theorem establishes, under certain hypotheses, the existence on some region of the phase space of canonical action–angle variables $(\varphi, I) = (\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$ in $\mathbb{T}^n \times G \subset \mathbb{T}^n \times \mathbb{R}^n$, in which the Hamiltonian only depends on the action variables: $h(I)$. The associated Hamiltonian equations for a trajectory $(\varphi(t), I(t))$ are

$$\dot{\varphi} = \omega(I), \qquad \dot{I} = 0,$$

where $\omega = \partial_I h$. Hence the dynamics is very simple: every n-dimensional torus $I =$ constant is invariant, with linear flow $\varphi(t) = \varphi(0) + \omega(I)t$, and thus all trajectories are stable. The motion on a torus is called quasiperiodic, with associated frequencies given by the vector $\omega(I) = (\omega_1(I), \dots, \omega_n(I))$.

Every n-dimensional invariant torus can be non-resonant or resonant, according to whether its frequencies are rationally independent or not. A non-resonant torus is densely filled by any of its trajectories. On the other hand, a resonant torus is foliated into a family of lower dimensional tori.

Figure: Non-resonant 2D Torus



A nearly-integrable Hamiltonian can be written in the form

$$H(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \qquad (1)$$

where ε is a small perturbation parameter. Then the Hamiltonian equations are

$$\dot{\varphi} = \omega(I) + \varepsilon \partial_I f(\varphi, I), \qquad \dot{I} = -\varepsilon \partial_\varphi f(\varphi, I).$$

For non-resonant, even more, Diophantine frequencies, KAM theorem provides n-dimensional invariant tori. For resonant frequencies there appear, typically, lower dimensional invariant tori, which are of saddle type, and that were called whiskered tori by Arnold because they have associated unstable and stable invariant manifolds.

Nekhoroshev theorem, first stated in 1977, establishes Effective stability for all the trajectories of a steep nearly-integrable system: For every initial condition $(\varphi(0), I(0))$ one has an estimate of the type

$$|I(t) - I(0)| \leq r_0\, \varepsilon^b \qquad \text{for } |t| \leq T_0 \exp\left\{(\varepsilon_0/\varepsilon)^a\right\}.$$

The constants $a, b > 0$ are called stability exponents [Nekhoroshev77] .

If h is quasiconvex, that is, for any $I \in G$ and $v \in \mathbb{R}^n$,

$$Dh(I)v = 0 \text{ and } v \neq 0 \quad \Longrightarrow v^\top D^2 h(I) v \neq 0,$$

the stability exponents are $a = b = \frac{1}{2n}$.

$$H^*(q,p,\varphi,I,s,A) = \frac{1}{2}\left(p^2 + I^2\right) + A + \varepsilon(\cos q - 1)\left(1 + \mu(\sin\varphi + \cos s)\right),$$

Since $h(p,I,A) = \frac{1}{2}\left(p^2 + I^2\right) + A$ satisfies $\begin{vmatrix} D^2h & Dh \\ Dh^\top & 0 \end{vmatrix} = -1 < 0$, one can check that h is quasiperiodic, and a priori

$$|(p,I,A)(t) - (p,I,A)(0)| \le r_0\,\varepsilon^{1/6} \qquad \text{for } |t| \le T_0 \exp\left\{(\varepsilon_0/\varepsilon)^{1/6}\right\}.$$

A refinement [Pöschel93, D-Gutiérrez96] for orbits close to the single resonance $p = 0$, using resonant normal forms, gives

$$|I(t) - I(0)| \le r_0\,\varepsilon^{1/4} \qquad \text{for } |t| \le T_0 \exp\left\{(\varepsilon_0/\varepsilon)^{1/4}\right\}.$$

For a nearly-integrable Hamiltonian with $n+1$ degrees of freedom

$$H(\varphi, I) = h(I) + \varepsilon f(\varphi, I), \qquad (\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$$

Select $I^* = 0$, and assume that the associated frequency vector $\lambda^* = \partial_I h(0) \in \mathbb{R}^{n+1}$ has a single resonance: $\langle k^*, \lambda^* \rangle = 0$ for some $0 \neq k^* \in \mathbb{Z}^{n+1}$ and $\langle k, \lambda^* \rangle \neq 0$ for any $k \in \mathbb{Z}^{n+1}$ not co-linear to k^*.
By a classical algebraic result, we can assume λ^* of the form

$$\lambda^* = (0, \omega^*)\,,$$

where $\omega^* \in \mathbb{R}^n$ is non-resonant. (If necessary, one can assume a Diophantine condition on ω^* to apply later a KAM theorem.)
The unperturbed Hamiltonian can be written (up to a constant) as:

$$h(I) = \langle \lambda^*, I \rangle + \frac{1}{2} \langle QI, I \rangle + \mathrm{O}_3(I).$$

Replace $\varphi \to (q, \varphi)$ and $I \to (p, I)$, and thus split $(\varphi, I) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ as $(q, p, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$, and the matrix $Q = \partial_I^2 h(0)$ as

$$\partial_{p,I}^2 h(0) = \begin{pmatrix} \beta^2 & \lambda^\top \\ \lambda & Q \end{pmatrix},$$

where we have put $\beta^2 > 0$ in order to fix ideas, $\lambda \in \mathbb{R}^n$ is a shift vector, and the new matrix Q is $n \times n$. We will assume $\beta = 1$; this can be achieved replacing p, I by p/β, I/β (changing in this way the time scale by a factor β), and rewriting ω^*/β, λ/β^2, Q/β^2 as ω^*, λ, Q respectively, and redefining also the function f.

Then, we can write our Hamiltonian in the form

$$H(q,p,\varphi,I) = h(p,I) + \varepsilon f(q,p,\varphi,I),$$
$$h(p,I) = \langle \omega^*, I\rangle + \frac{p^2}{2} + \langle \lambda, I\rangle\, p + \frac{1}{2}\langle QI, I\rangle + \mathrm{O}_3(p,I).$$

We now perform one step of resonant normal form procedure: following the Lie method, we seek for functions $S(q,\varphi)$ and $R(q,p,\varphi,I) = \mathrm{O}(p,I)$ such that

$$\{S,h\} + V + R = f, \qquad (2)$$

where $V(q)$ is the periodic function obtained by averaging $f(q,0,\varphi,0)$ with respect to the angles φ:

$$V(q) = \overline{f(q,0,\cdot,0)} = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(q,0,\varphi,0)\mathrm{d}\varphi, \qquad q \in \mathbb{T}.$$

The construction of S and R is easily carried out: assuming a Diophantine condition on ω^*, one first solves the equation

$$\langle \omega^*, \partial_\varphi S \rangle + V = f(\cdot, 0, \cdot, 0)$$

with the help of standard small divisors estimates, and then one takes R simply by fitting equation (2). The time-1 symplectic flow Φ of the generating Hamiltonian εS leads to

$$H \circ \Phi = H + \{H, \varepsilon S\} + \mathrm{O}\left(\varepsilon^2\right) = h + \varepsilon(V + R) + \mathrm{O}\left(\varepsilon^2\right) = H_0 + H_1,$$

with

$$H_0(q,p,I;\varepsilon) = \langle \omega^*, I \rangle + \frac{p^2}{2} + \varepsilon V(q) + \langle \lambda, I \rangle\, p + \frac{1}{2}\langle QI, I \rangle\,,$$
$$H_1(q,p,\varphi,I;\varepsilon) = \varepsilon R(q,p,\varphi,I) + \mathrm{O}_3(p,I) + \mathrm{O}\left(\varepsilon^2\right).$$

Note: $\omega^* = \lambda = 0$, $V(q) = \cos q - 1$, $H_1 = \mathrm{O}(\varepsilon\mu)$ in the Arnold example.

This expression generalizes Arnold's example.
Concerning V, except for degenerate cases, the function $V(q)$ will have a unique and nondegenerate maximum q_0; we denote $\alpha^2 = -V''(q_0) > 0$.
Then, for $\varepsilon > 0$, the 1-degree-of-freedom Hamiltonian

$$P(q, p; \varepsilon) = \frac{p^2}{2} + \varepsilon V(q),$$

has a saddle point in $(q_0, 0)$, with (homoclinic) separatrices. The case $\varepsilon < 0$ is analogous, provided one considers a minimum instead of a maximum. Then, the Hamiltonian H_0 has whiskered tori with coincident whiskers associated to this saddle point.
Note that H_0 constitutes a Hamiltonian situated between the unperturbed Hamiltonian h and the perturbed one H, which possesses hyperbolic invariant tori but their whiskers still coincide.
Note also that, in general, H_0 is not an uncoupled Hamiltonian because of the coupling term $\langle \lambda, I \rangle \, p$.

The Lyapunov exponents of the saddle point of the "pendulum" P are $\pm\sqrt{\varepsilon}\alpha$, which tend to zero for $\varepsilon \to 0^+$.
To have fixed Lyapunov exponents, we can replace p, I by $\sqrt{\varepsilon}p$, $\sqrt{\varepsilon}I$.
The new system is still Hamiltonian if we divide the Hamiltonian by ε (making in this way a change of time scale by a factor $\sqrt{\varepsilon}$):

$$H_0 = \langle \omega, I \rangle + \frac{p^2}{2} + V(q) + \langle \lambda, I \rangle p + \frac{1}{2} \langle QI, I \rangle , \quad (3)$$

$$H_1 = R\left(x, \sqrt{\varepsilon}y, \varphi, \sqrt{\varepsilon}I\right) + \frac{1}{\varepsilon} \mathrm{O}_3\left(\sqrt{\varepsilon}y, \sqrt{\varepsilon}I\right) + \mathrm{O}\left(\varepsilon\right) = \mathrm{O}(\mu), \quad (4)$$

where

$$\omega = \frac{\omega^*}{\sqrt{\varepsilon}}, \qquad \mu = \sqrt{\varepsilon}.$$

For $\varepsilon \to 0^+$, the study of the Hamiltonian (3–4) is a singular perturbation problem, due to the fast frequencies $\omega = \omega^*/\sqrt{\varepsilon}$ in the unperturbed Hamiltonian H_0. We are thus confronted with a singular system, often referred to as weakly hyperbolic, and also called a-priori stable [Chierchia-Gallavotti94] . In fact, this case can be referred to as totally singular, because all the frequencies are fast.

The singular problem can be avoided if one considers independent parameters, namely a fixed $\varepsilon > 0$ (that is, a fixed ω in (3)) and $\mu \to 0$. In such a case, the system (3–4) has the property that the hyperbolicity and the homoclinic orbits are present in the unperturbed Hamiltonian ($\mu = 0$), and are simply perturbed for $|\mu|$ small. In this case, we are confronted with a regular or strongly hyperbolic system, or also a-priori unstable.

This strategy of keeping $\varepsilon > 0$ fixed and letting $\mu \to 0$ was introduced by Poincaré in 1889 and followed in Arnold's example to avoid dealing with a singular perturbation problem.

Unfortunately, the exponentially small splitting of separatrices predicted by a direct application of the Poincaré-Arnold-Melnikov (PMA) method

$$\text{Splitting distance} = \varepsilon \text{ PMA prediction} + \mathrm{O}(\varepsilon\mu)$$

when the PMA prediction $= \mathrm{O}\left(e^{-c/\varepsilon^{a}}\right)$ could then be justified only for μ exponentially small in ε.

$$H(q,p,\varphi,I,s) = \frac{1}{2}p^2 + \varepsilon(\cos q - 1) + \frac{1}{2}I^2 + \varepsilon\mu f(q) g(\varphi,s)$$

$$f(q) = \cos q - 1, \qquad g(\varphi,s) = \sin\varphi + \cos s,$$

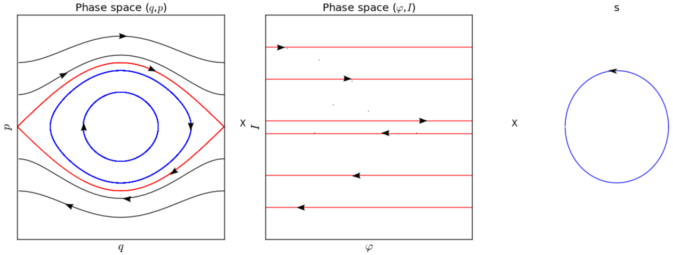


Figure: Phase Space - Unperturbed problem for $\varepsilon = 0$

- Invariant tori (2D)

$$\widetilde{\mathcal{T}}_I = \{(0,0,I,\varphi,s) : (\varphi,s) \in \mathbb{T}^2\}$$

- Invariant manifolds (3D):

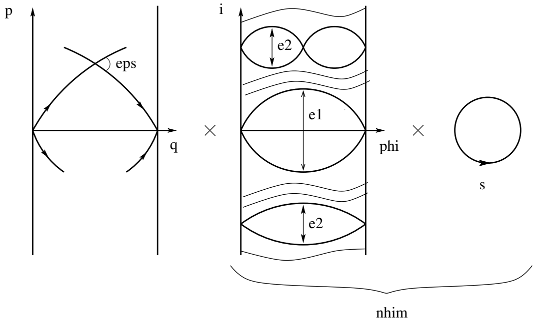
$$W^s\widetilde{\mathcal{T}}_I = W^u\widetilde{\mathcal{T}}_I = \{(q_0(\sqrt{\varepsilon}\tau), \sqrt{\varepsilon}p_0(\sqrt{\varepsilon}\tau), I, \varphi, s) : \tau \in \mathbb{R}, (\varphi,s) \in \mathbb{T}^2\}$$

where

$$q_0(t) = 4\arctan \mathrm{e}^{\pm t}, \quad p_0(t) = 2/\cosh t.$$

is the separatrix for positive p of the standard pendulum

$$P(q,p) = p^2/2 + \cos q - 1.$$



- By the special form of the perturbation, $\widetilde{\mathcal{T}}_I$ persist to $\widetilde{\mathcal{T}}_I^{\varepsilon} = \widetilde{\mathcal{T}}_I$
- $W^s\widetilde{\mathcal{T}}_I^{\varepsilon}$ and $W^u\widetilde{\mathcal{T}}_I^{\varepsilon}$ are ε-close to the unperturbed ones.
- Using Poincaré-Melnikov theory, $W^s\widetilde{\mathcal{T}}_I^{\varepsilon} \pitchfork W^u\widetilde{\mathcal{T}}_I^{\varepsilon}$ with an angle of size $e^{-\pi/(2\sqrt{\varepsilon})}$.
- Therefore $W^s\widetilde{\mathcal{T}}_{I_i}^{\varepsilon} \pitchfork W^u\widetilde{\mathcal{T}}_{I_{i+1}}^{\varepsilon}$ for $|I_i - I_{i+1}| \leq e^{-\pi/(2\sqrt{\varepsilon})}$ and a shadowing (transition chain mechanism) gives the diffusion path.

Minor: 4 pages paper in Dokl. Akad. Nauk SSSR. "The details of the proof must be formidable, although the idea of the proof is clearly outlined." (J. Moser in the MathSciNet review)

Fixable: The perturbation maintains fixed all the invariant tori $\mathcal{T}_I$. In general, there appear gaps around resonant tori (rational I) which prevent $W^s\widetilde{\mathcal{T}}^{\varepsilon}_{I_i} \pitchfork W^u\widetilde{\mathcal{T}}^{\varepsilon}_{I_{i+1}}$ because $\widetilde{\mathcal{T}}^{\varepsilon}_{I_i}$ and $\widetilde{\mathcal{T}}^{\varepsilon}_{I_{i+1}}$ are too far. The Scattering map can fix it.

Major: The exponentially small size of the splitting $\mathrm{e}^{-\pi/(2\sqrt{\varepsilon})}$ computed from a direct application of the PMA method is much less than the Nekhoroshev estimates $\mathrm{e}^{-\pi/(2\varepsilon^{1/4})}$.

Major: Arnold example only shows global instability along a single resonance, where the associated normal form is integrable, but does not deal with multiple resonances, where the normal form is not integrable.

Exponentially small splitting of separatrices

The exponentially small splitting of separatrices was already found by Poincaré in 1890, and first addressed in 1984 by Neishtadt with upper bounds using normal forms and by Lazutkin with asimptotic estimates using complex parameterizations of the stable and unstable invariant manifolds.

Proofs of its asymptotic behavior for the rapidly forced pendulum or other rapidly oscillating periodic perturbations in [D-Seara92,Gelfreich94,Fontich93-95,Sauzin95,Treschev97,D-Seara97,Gelfreich97,Baldomá-Fontich04-05,Guardia-Olivé-Seara10,Baldomá-Fontich-Guardia-Seara12]

For maps, upper exponentially small estimates in [Fontich-Simó90]] and asymptotic estimates in [D-Ramírez-Ros98-99,Simó-Vieiro09,Martín-Sauzin-Seara11]

Exponentially small splitting of separatrices

In the rapidly quasiperiodically forced pendulum, the rôle of the arithmetic properties was detected in [Simó94] , and established in [D-Gelfreich-Seara-Jorba97] .

For n-dimensional whiskered tori of a Hamiltonian with $n+1$ degrees of freedom, the splitting potential and Melnikov potential were introduced [Eliasson94,D-Gutiérrez00] , sharp exponentially small upper bounds were given in [D-Gutiérrez-Seara04] , and asymptotic estimates in [Lochak-Marco-Sauzin03,D-Gutiérrez04,D-GonchenkoGutiérrez14-16] .

The multidimensional separatrix map introduced by Treschev in 2002 requires more study.

We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\varepsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \\ &= P_\pm(p,q) + \tfrac{1}{2}I^2 + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \end{aligned} \tag{5}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{6}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the pendulum.

Note. This model just comes from the single resonance normal form. The perturbation is arbitrary.

Theorem (D-Llave-Seara06)

Consider the Hamiltonian (5) where V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume also that

H1 *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*

H2 *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):*

$$\begin{aligned} \mathcal{L}(I, \varphi, s) &= -\int_{-\infty}^{+\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) \\ &\qquad -h(0, 0, I, \varphi + I\sigma, s + \sigma; 0))d\sigma \end{aligned} \tag{7}$$

satisfies concrete non-degeneracy conditions.

H3 *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system (5) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

Remark Arbitrary excursions in the I variable can also be realized.

Hypotheses **H1**, **H2** and **H3** are $\mathcal{C}^2$ generic, so, the following short version of the Theorem also holds:

Theorem (D-Huguet09)

Consider the Hamiltonian (5) and assume that V and h are $\mathcal{C}r+2$ functions which are $\mathcal{C}^2$ generic, with $r > r_0$, large enough. Then there is $\varepsilon^* > 0$ such that for $0 < |\varepsilon| < \varepsilon^*$ and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\tilde{x}(t)$ of the system with Hamiltonian (5) such that for some $T > 0$

$$I(\tilde{x}(0)) \leq I_-^*; \qquad I(\tilde{x}(T)) \geq I_+^*.$$

Remark A (non optimal) value of r_0 which follows from our argument is $r_0 = 242$.

Consider a periodic in time perturbation of n pendula and a d-dimensional rotor described by the non-autonomous Hamiltonian,

$$H(p,q,I,\varphi,t,\varepsilon) = P(p,q) + h(I) + \varepsilon Q(p,q,I,\varphi,t,\varepsilon), \tag{8}$$

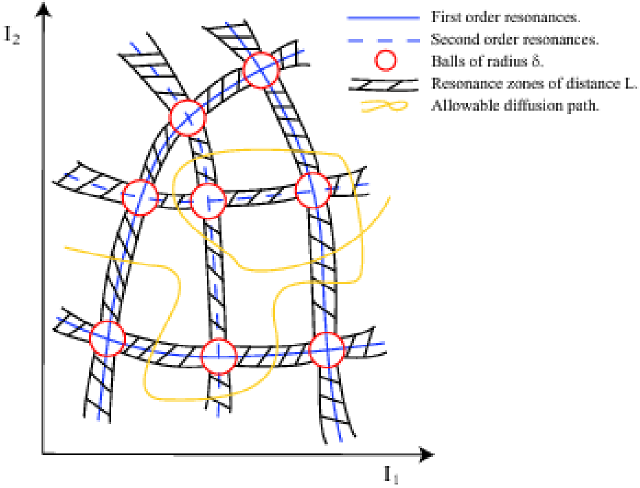
with $P(p,q) = \sum_{j=1}^{n} P_j(p_j,q_j)$, $P_j(p_j,q_j) = \pm\left(\frac{1}{2}p_j^2 + V_j(q_j)\right)$, where $I \in \mathcal{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $\mathcal{I}$ an open set, $p,q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$, and $P_j(p_j,q_j)$ is a pendulum for the saddle variables p_j, q_j. For $\varepsilon = 0$, the d-dimensional action I remains constant. Under similar hypotheses as for $n = d = 1$,

Theorem (D-Llave-Seara12)

For every $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for every $0 < |\varepsilon| < \varepsilon_0$, given $I_\pm \in \mathcal{I}$, there exists a solution $\tilde{x}(t)$ of (8) and $T > 0$, such that

$$|I(\tilde{x}(0)) - I_-| \le C\delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \le C\delta \tag{9}$$

- One can forget about δ and prescribe arbitrary paths on a set $\mathcal{I}^*$. This set $\mathcal{I}^*$ is described precisely in the course of the proof, and is determined by the non-degeneracy assumptions. The main idea is that $\mathcal{I}^*$ is obtained from the domain of definition, just eliminating some sets of codimension 2, like double resonances, from the open set where the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is transversal.
- Codimension 2 objects do not separate the regions and can be contoured so that they do not obstruct the change along the paths. It seems that such contouring trajectories close to double resonances are inferred from some movies related to numerical experiments in [Gelfreich-Simó-Vieiro13] .



I_2
I_1
First order resonances.
Second order resonances.
Balls of radius δ.
Resonance zones of distance L.
Allowable diffusion path.

Other contributions

This problem of instability, also called Arnold diffusion, was posed first by Arnold in 1964, and there have been some other contributions, using geometrical or variational methods:
[Lochak92] , [Chierchia-Gallavotti94-98] , [Bessi-Chierchia-Valdinoci01] [Berti-Biasco-Bolle03] , [Marco-Sauzin03] , [Mather04] , [Cheng-Yan04] , [Gidea-Llave06] , [Piftankin-Treschev07] , [Kaloshin-Levi08] , [ChengY09] , [Bernard-Kaloshin-Zhang16] , [Zhang11] , [Mather12] , [Treschev12] , [Gelfreich-Simó-Vieiro13] , [GelfreichT17] , [Gidea-Llave-Seara14] , [Kaloshin-Zhang15] , [Lazzarini-Marco-SauzinS15] , [Davletshin-Treschev16] , [Marco16] , [Gidea-Marco17] , [Cheng17] .

The main idea of the proof is to use the two (or more) dynamics on $\widetilde{\Lambda}$.

- Find a big invariant saddle object: a NHIM (normally hyperbolic invariant manifold: a global version of a center manifold) $\widetilde{\Lambda}$ with transverse associated stable and unstable manifolds along some homoclinic manifold Γ: $\mathcal{W}^u(\widetilde{\Lambda}) \pitchfork_\Gamma \mathcal{W}^s(\widetilde{\Lambda})$.
- Compute the invariant objects (typically tori $\mathcal{T}$) which may prevent instability for the inner dynamics of the NHIM.
- Compute an scattering map $S = S^\Gamma : H_- \subset \widetilde{\Lambda} \to H_+ \subset \widetilde{\Lambda}$ on the NHIM associated to Γ and consider it as an outer dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a transition chain of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the transition chain.

Consider a pendulum and a rotor plus a time periodic perturbation depending on two harmonics in the variables (φ, s):

$$H_\varepsilon(p,q,I,\varphi,s) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \varepsilon h(q,\varphi,s) \tag{10}$$

$$\begin{gathered} h(q,\varphi,s) = f(q)g(\varphi,s), \qquad f(q) = \cos q, \\ g(\varphi,s) = a_1\cos(k_1\varphi + l_1 s) + a_2\cos(k_2\varphi + l_2 s), \end{gathered} \tag{11}$$

for some $k_1,\ k_2,\ l_1,\ l_2 \in \mathbb{Z}$.

Theorem

Assume that $a_1 a_2 \neq 0$ *and* $\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0$ *in* (10)-(11). *Then, for any* $I^* > 0$, *there exists* $\varepsilon^* = \varepsilon^*(I^*, a_1, a_2) > 0$ *such that for any* ε, $0 < \varepsilon < \varepsilon^*$, *there exists a trajectory* $(p(t), q(t), I(t), \varphi(t))$ *such that for some* $T > 0$

$$I(0) \leq -I^* < I^* \leq I(T).$$

Remark: $I(t) \equiv$ constant for $\varepsilon = 0$.

- To review the construction of scattering maps initiated in [D-Llave-Seara00] , designed to detect global instability.
- To compute explicitly several scattering maps to prove global instability for the action I for any $\varepsilon > 0$ small enough.
- To estimate the time of diffusion in some cases (at least for $k_1 = l_2 = 1$ and $l_1 = k_2 = 0$).
- To play with the parameter $\mu = a_1/a_2$ to prove global instability for any value of $\mu \neq 0, \infty$.
- To describe bifurcations of the scattering maps.
- To get a glimpse of the $3 + \frac{1}{2}$ degrees of freedom case.

It is easy to check that if

$$\Delta := k_1 l_2 - k_2 l_1 = 0 \quad \text{or} \quad a_1 = 0 \quad \text{or} \quad a_2 = 0$$

there is no global instability for the variable I.

If $\Delta a_1 a_2 \neq 0$, after some rational linear changes in the angles, we only need to study two cases:

- The first (and easier) case [D-Schaefer17]

$$g(\varphi, s) = a_1 \cos\varphi + a_2 \cos s$$

- The second case [D-Schaefer18]

$$g(\varphi, \sigma) = a_1 \cos\varphi + a_2 \cos\sigma,$$

where $\sigma = \varphi - s$.

We deal with an a priori unstable Hamiltonian [Chierchia-Gallavotti94] .

In the unperturbed case $\varepsilon = 0$, the Hamiltonian H_0 is integrable formed by the standard pendulum plus a rotor

$$H_0(p, q, I, \varphi, s) = \pm \left(\frac{p^2}{2} + \cos q - 1 \right) + \frac{I^2}{2}.$$

I is constant: $\quad \triangle I := I(T) - I(0) \equiv 0.$

For any $0 < \varepsilon \ll 1$, there is a finite drift in the action of the rotor I: $\triangle I = \mathcal{O}(1)$, so we have global instability.

In short, this is is also frequently called Arnold diffusion.

Basically, we ensure the Arnold diffusion performing the following scheme:

- To construct iterates under several Scattering maps and the Inner map, giving rise to diffusing pseudo-orbits.
- To use previous results about Shadowing [Fontich-Martín00] , [Gidea-Llave-Seara14] , for ensuring the existence of real orbits close to the pseudo-orbits.

We have two important dynamics associated to the system: the inner and the outer dynamics on a large invariant object $\widetilde{\Lambda}$:

$$\widetilde{\Lambda} = \{(0,0,I,\varphi,s); I \in [-I^*, I^*], (\varphi,s) \in \mathbb{T}^2\},$$

which is a 3D Normally Hyperbolic Invariant Manifold (NHIM) with associated 4D stable $W^s_\varepsilon(\widetilde{\Lambda})$ and unstable $W^u_\varepsilon(\widetilde{\Lambda})$ invariant manifolds.

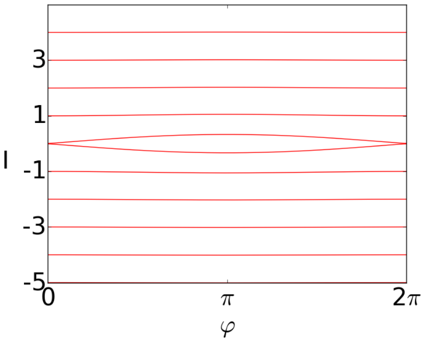
- The inner dynamics is the dynamics restricted to $\widetilde{\Lambda}$. (Inner map)
- The outer dynamics is the dynamics along the invariant manifolds to $\widetilde{\Lambda}$. (Scattering map)

Remark: Due to the form of the perturbation, $\widetilde{\Lambda} = \widetilde{\Lambda}_\varepsilon$.

For the first case $g(\varphi, s) = a_1 \cos\varphi + a_2 \cos s$, the inner dynamics is described by the Hamiltonian systems with the Hamiltonian

$$K(I, \varphi, s) = \frac{I^2}{2} + \varepsilon \left(a_1 \cos\varphi + a_2 \cos s\right).$$

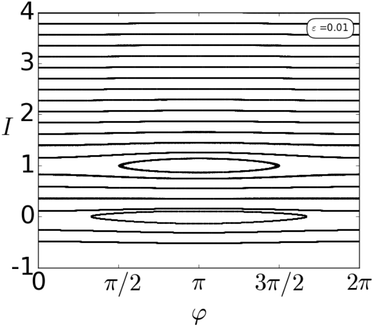
In this case the inner dynamics is integrable.



For $g(\varphi,\sigma)$, the inner dynamics is by the Hamiltonian

$$K(I,\varphi,\sigma)=\frac{I^2}{2}+\varepsilon\left(a_1\cos\varphi+a_2\cos\sigma\right),$$

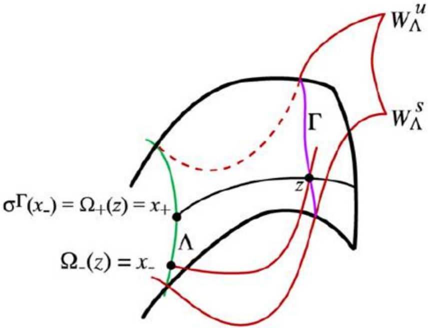
where $\sigma=\varphi-s$. The system associated to this Hamiltonian is not integrable and two resonances arise in $I=0$ and $I=1$.



Let $\widetilde{\Lambda}$ be a NHIM with invariant manifolds intersecting transversally along a homoclinic manifold Γ. A scattering map is a map S defined by $S(\tilde{x}_-) = \tilde{x}_+$ if there exists $\tilde{z} \in \Gamma$ satisfying

$$|\phi_t^\varepsilon(\tilde{z}) - \phi_t^\varepsilon(\tilde{x}_\mp)| \longrightarrow 0 \text{ as } t \longrightarrow \mp\infty$$

that is, $W_\varepsilon^u(\tilde{x}_-)$ intersects transversally $W_\varepsilon^s(\tilde{x}_+)$ in $\tilde{z}$.



S is symplectic and exact [D-Llave-Seara08] and takes the form:

$$S_\varepsilon(I,\varphi,s) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2), s \right),$$

where $\theta = \varphi - Is$ and $\mathcal{L}^*(I,\theta)$ is the Reduced Poincaré function, or more simply in the variables (I,θ):

$$\mathcal{S}_\varepsilon(I,\theta) = \left(I + \varepsilon \frac{\partial \mathcal{L}^*}{\partial \theta}(I,\theta) + \mathcal{O}(\varepsilon^2), \theta - \varepsilon \frac{\partial \mathcal{L}^*}{\partial I}(I,\theta) + \mathcal{O}(\varepsilon^2) \right),$$

- The variable s remains fixed under S_ε: it plays the role of a parameter
- Up to first order in ε, $\mathcal{S}_\varepsilon$ is the $-\varepsilon$-time flow of the Hamiltonian $\mathcal{L}^*(I,\theta)$
- The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I,\theta)$

To get a scattering map we search for homoclinic orbits to $\tilde{\Lambda}_\varepsilon$

Proposition

Given $(I, \varphi, s) \in [-I^*, I^*] \times \mathbb{T}^2$, assume that the real function

$$\tau \in \mathbb{R} \longmapsto \mathcal{L}(I, \varphi - I\,\tau, s - \tau) \in \mathbb{R}$$

has a non degenerate critical point $\tau^* = \tau(I, \varphi, s)$, where

$$\mathcal{L}(I, \varphi, s) = \int_{-\infty}^{+\infty} (\cos q_0(\sigma) - \cos 0)\, g(\varphi + I\sigma, s + \sigma; 0) d\sigma.$$

Then, for $0 < |\varepsilon|$ small enough, there exists a transversal homoclinic point $\tilde{z}$ to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to the point $\tilde{z}^*(I, \varphi, s) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) \in W^0(\widetilde{\Lambda})$:

$$\tilde{z} = \tilde{z}(I, \varphi, s) = (p_0(\tau^*) + O(\varepsilon), q_0(\tau^*) + O(\varepsilon), I, \varphi, s) \in W^u(\widetilde{\Lambda}_\varepsilon) \pitchfork W^s(\widetilde{\Lambda}_\varepsilon).$$

In our model $q_0(t) = 4 \arctan \mathrm{e}^t$, $p_0(t) = 2/\cosh t$ is the separatrix for positive p of the standard pendulum $P(q, p) = p^2/2 + \cos q - 1$.

- For $g(\varphi, s) = a_1 \cos \varphi + a_2 \cos s$, the Melnikov potential becomes

$$\mathcal{L}(I, \varphi, s) = A_1(I) \cos \varphi + A_2 \cos s,$$

where $A_1(I) = \dfrac{2\,\pi\, I\, a_1}{\sinh\left(\frac{I\pi}{2}\right)}$ and $A_2 = \dfrac{2\,\pi\, a_2}{\sinh\left(\frac{\pi}{2}\right)}$.

- For $g(\varphi, \sigma) = a_1 \cos \varphi + a_2 \cos \sigma$ $(\sigma = \varphi - s)$, the Melnikov potential becomes

$$\mathcal{L}(I, \varphi, \sigma) = A_1(I) \cos \varphi + A_2(I) \cos \sigma,$$

where $A_1(I)$ is as before but now $A_2(I) = \dfrac{2\,(I-1)\pi\, a_2}{\sinh\left(\frac{(I-1)\pi}{2}\right)}$.

The Melnikov potentials are similar in both cases.

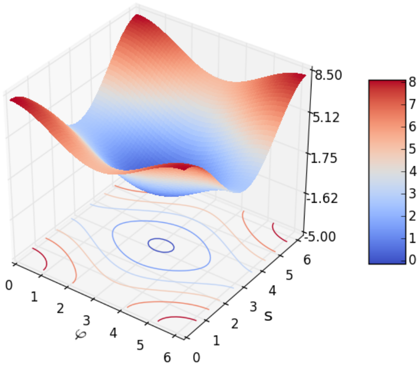


Figure: The Melnikov Potential, $\mu = a_1/a_2 = 0.6$, $I = 1$, $g(\varphi, s)$.

Finally, the function $\mathcal{L}^*(I,\theta)$ can be defined:

Definition

The Reduced Poincaré function is

$$\mathcal{L}^*(I,\theta) = \mathcal{L}(I, \varphi - I\,\tau^*(I,\varphi,s), s - \tau^*(I,\varphi,s)),$$

where $\theta = \varphi - I\,s$.

Therefore the definition of $\mathcal{L}^*(I,\theta)$ depends on the function $\tau^*(I,\varphi,s)$.

From the Proposition given above, we look for τ^* such that $\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\,\tau^*, s - \tau^*) = 0$.

Different view-points for $\tau^* = \tau^*(I, \varphi, s)$

- Look for critical points of $\mathcal{L}$ on the straight line, called NHIM line $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$.
- Look for intersections between $R(I, \varphi, s) = \{(\varphi - I\,\tau, s - \tau),\ \tau \in \mathbb{R}\}$ and a crest which is a curve of equation

$$\frac{\partial \mathcal{L}}{\partial \tau}(I, \varphi - I\tau, s - \tau)\big|_{\tau=0} = 0.$$

Note that the crests are characterized by $\tau^*(I, \varphi, s) = 0$.

Definition - Crests [D-Huguet11]

For each I, we call crest $\mathcal{C}(I)$ the set of curves in the variables (φ, s) of equation

$$I\frac{\partial \mathcal{L}}{\partial \varphi}(I,\varphi,s)+\frac{\partial \mathcal{L}}{\partial s}(I,\varphi,s)=0. \tag{12}$$

which in our case can be rewritten as

$$g(\varphi,s):\ \ \mu\alpha(I)\sin\varphi+\sin s=0, \qquad \text{with } \alpha(I)=\frac{I^2\sinh(\frac{\pi}{2})}{\sinh(\frac{\pi I}{2})}, \quad \mu=\frac{a_1}{a_2}.$$

$$g(\varphi,\sigma=\varphi-s):\ \ \mu\alpha(I)\sin\varphi+\sin\sigma=0, \qquad \text{with } \alpha(I)=\frac{I^2\sinh(\frac{(I-1)\pi}{2})}{(I-1)^2\sinh(\frac{\pi I}{2})}, \quad \mu=\frac{a_1}{a_2}.$$

- For any I, the critical points of the Melnikov potential $\mathcal{L}(I,\cdot,\cdot)$ ($(0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π): one maximum, one minimum point and two saddle points) always belong to the crest $\mathcal{C}(I)$.
- $\mathcal{L}^*(I,\theta)$ is nothing else but $\mathcal{L}$ evaluated on the crest $\mathcal{C}(I)$.
- $\theta=\varphi-Is$ is constant on the NHIM line $R(I,\varphi,s)$

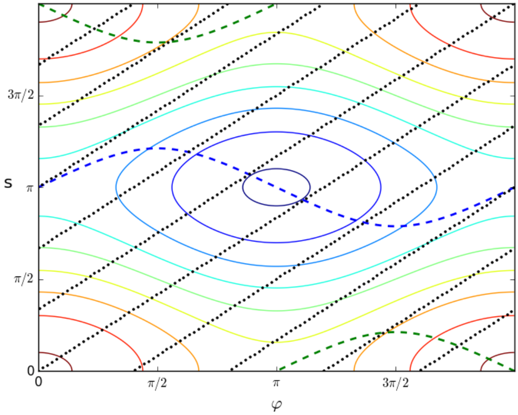


Figure: Level curves of $\mathcal{L}$ for $\mu = a_1/a_2 = 0.5$, $I = 1.2$ and $g(\varphi, s)$.

Understanding the behavior of the crests

$\Downarrow$

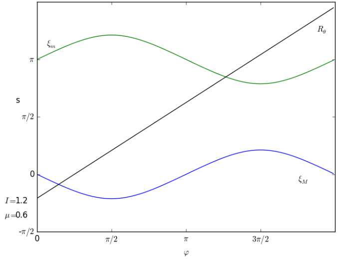
Understanding the behavior of the Reduced Poincaré function

$\Downarrow$

Understanding the Scattering map

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by:

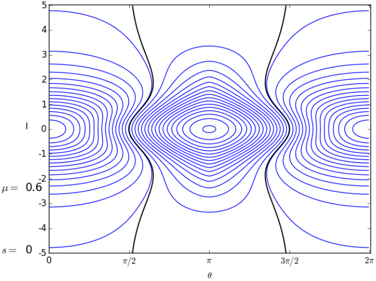
$$\begin{aligned} s = \xi_M(I,\varphi) &= -\arcsin(\mu\alpha(I)\sin\varphi) \mod 2\pi \\ \xi_m(I,\varphi) &= \arcsin(\mu\alpha(I)\sin\varphi) + \pi \mod 2\pi \end{aligned} \quad (13)$$



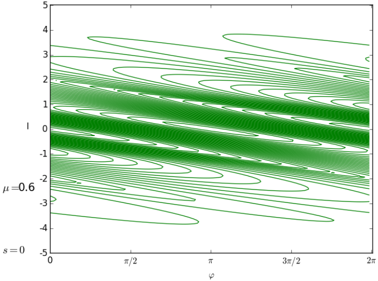
They are "horizontal" crests

First case: $g(\varphi, s)$ $\quad 0 < |\mu| < 0.625$

- For each I, the NHIM line $R(I, \varphi, s)$ and the crest $\mathcal{C}_{\mathrm{M,m}}(I)$ has only one intersection point.
- The scattering map S_{M} associated to the intersections between $\mathcal{C}_{\mathrm{M}}(I)$ and $R(I, \varphi, s)$ is well defined for any $\varphi \in \mathbb{T}$. Analogously for S_{m}, changing M to m. In the variables $(I, \theta = \varphi - Is)$, both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.

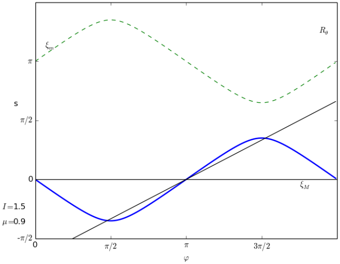


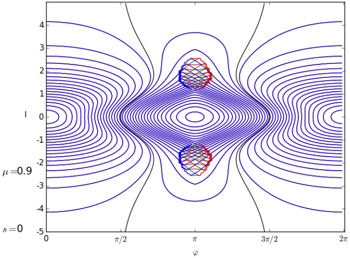
(a) Level curves of $\mathcal{L}^*_M(I, \theta)$



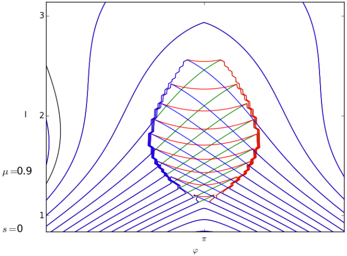
(b) Level curves of $\mathcal{L}^*_m(I, \theta)$

- There are tangencies between $\mathcal{C}_{\mathrm{M,m}}(I,\varphi)$ and $R(I,\varphi,s)$. For some value of (I,φ,s), there are 3 points in $R(I,\varphi,s)\cap\mathcal{C}_{\mathrm{M,m}}(I)$.
- This implies that there are 3 scattering maps associated to each crest with different domains.(Multiple Scattering maps)





(c) The three types of level curves.

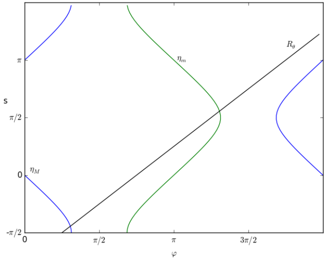


(d) Zoom where the scattering maps are different

Figure: Level curves of $\mathcal{L}_M^*(I,\theta)$, $\mathcal{L}_M^{*(1)}(I,\theta)$ and $\mathcal{L}_M^{*(2)}(I,\theta)$

- For some values of I, $|\mu\alpha(I)| > 1$, the two crests $\mathcal{C}_{\mathrm{M,m}}$ are parameterized by:

$$\begin{aligned} \varphi = \eta_M(I,s) &= -\arcsin(\mu\alpha(I)\sin s) \mod 2\pi \\ \eta_m(I,s) &= \arcsin(\mu\alpha(I)\sin s) + \pi \mod 2\pi \end{aligned} \tag{14}$$



They are "vertical" crests

For the values of I for which horizontal crests become vertical, it is not always possible to prolong in a continuous way the scattering maps, so the domain of the scattering map has to be restricted.

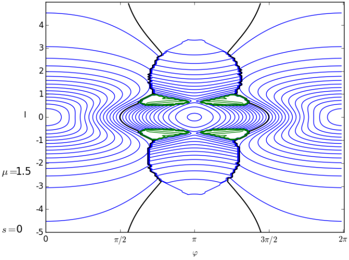


Figure: The level curves of $\mathcal{L}^*_{\mathrm{M}}(I, \theta)$, $\mu = 1.5$.

In green, the region where the scattering map S_{M} is not defined.

Definition: Highways

Highways are the level curves of $\mathcal{L}^*$ such that

$$\mathcal{L}^*(I, \theta) = \frac{2\pi a_1}{\sinh(\pi/2)}.$$

- The highways are "vertical" in the variables (φ, s)
- We always have a pair of highways. One goes up, the other goes down (this depends on the sign of $\mu = a_1/a_2$)
- The highways give rise to fast diffusing pseudo-orbits

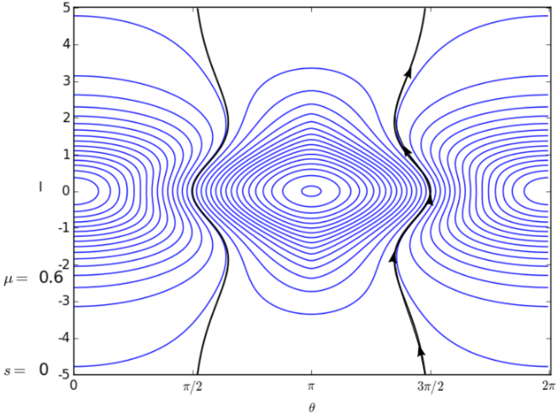


Figure: The scattering map jumps $\mathcal{O}(\varepsilon)$ distances along the level curves of $\mathcal{L}^*(I, \theta)$

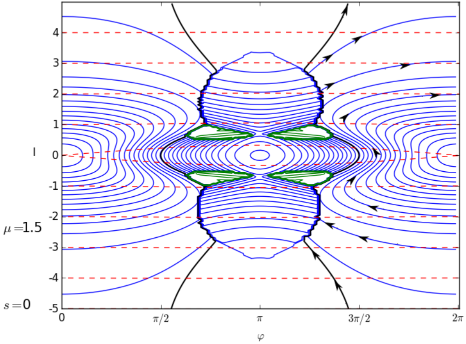


Figure: In red: Inner map, blue: Scattering map, black: Highways

An estimate of the total time of diffusion between $-I^*$ and I^*, along the highway, is

$$T_{\mathrm{d}} = \frac{T_{\mathrm{s}}}{\varepsilon}\left[2\log\left(\frac{C}{\varepsilon}\right) + \mathcal{O}(\varepsilon^b)\right], \text{ for } \varepsilon \to 0, \text{ where } 0 < b < 1,$$

with

$$T_{\mathrm{s}} = T_{\mathrm{s}}(I^*, a_1, a_2) = \int_0^{I^*} \frac{-\sinh(\pi I/2)}{\pi a_1 I \sin\psi_{\mathrm{h}}(I)}\, dI,$$

where $\psi_{\mathrm{h}} = \theta - I\tau^*(I,\theta)$ is the parameterization of the highway $\mathcal{L}^*(I,\psi_{\mathrm{h}}) = A_2$, and

$$C = C(I^*, a_1, a_2) = 16\,|a_1|\left(1 + \frac{1.465}{\sqrt{1-\mu^2 A^2}}\right)$$

where $A = \max_{I\in[0,I^*]} \alpha(I)$, with $\alpha(I) = \frac{\sinh(\frac{\pi}{2})\, I^2}{\sinh(\frac{\pi I}{2})}$ and $\mu = a_1/a_2$.

Note: This estimate agrees with the upper bounds given in [Bessi-Chierchia-Valdinoci01] and quantifies the general optimal diffusion estimate $\mathcal{O}\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ of [Berti-Biasco-Bolle03] and [Treschev04] .

In the second case:

- For $|\mu\alpha(I)| < 1$, there are two crests $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by $\sigma = \xi_M(I, \varphi)$ and $\xi_m(I, \varphi)$. For $|\mu\alpha(I)| > 1$, $\mathcal{C}_{\mathrm{M,m}}(I)$ parameterized by $\varphi = \eta_M(I, \sigma)$ and $\eta_m(I, \sigma)$. The crests lie on the plane (φ, σ)

- There are no Highways.

- For any value of $\mu = a_1/a_2$ is possible to find I_{h} and I_{v} such that for $I = I_{\mathrm{h}}$ the crests are horizontal and for $I = I_{\mathrm{v}}$ the crests are vertical.

- For any value of μ there exists I such that the crests and some NHIM line are tangent. There are always multiple scattering maps

From the definitions of $R(I, \varphi, s)$ and $\mathcal{C}(I)$, we have

$$R(I, \varphi, s) \cap \mathcal{C}(I) = \{(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s))\}.$$

Introducing

$$\tau^*(I, \theta) := \tau^*(I, \varphi - Is), \qquad \text{with } \theta = \varphi - Is = (1 - I)\varphi + I\sigma,$$

one can see that on the plane $(\varphi, \sigma = \varphi - s)$, the NHIM lines take the form

$$R_I(\varphi, \sigma) = \{(\varphi - I\tau, \sigma - (I - 1)\tau), \tau \in \mathbb{R}\}$$

and that

$$R_I(\varphi, \sigma) \cap \mathcal{C}(I) = \{(\theta - I\tau^*(I, \theta), \theta - (I - 1)\tau^*(I, \theta))\}.$$

Therefore, the function $\tau^*(I, \theta)$ is the time spent to go from a point (θ, θ) in the diagonal $\sigma = \varphi$ up to $\mathcal{C}(I)$ with a velocity vector $\mathbf{v} = -(I, I - 1)$.

The choice of the concrete curve of the crest and therefore of $\tau^*(I, \theta)$ is very important and useful.

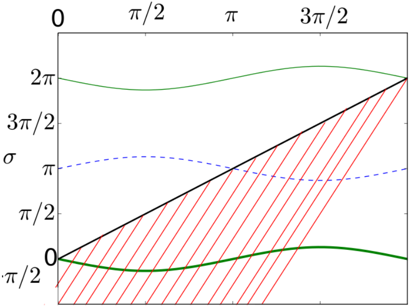


Figure: Going down along NHIM lines

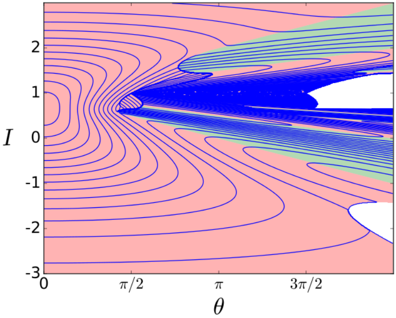


Figure: The "lower" crest

Green zones: I increases under the scattering map.

Red zones: I decreases under the scattering map.

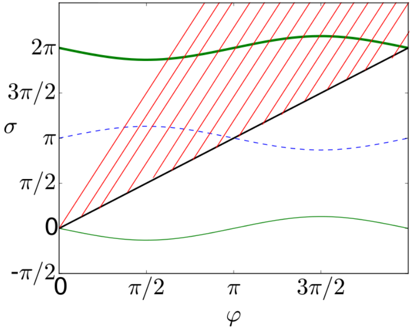


Figure: Going up along NHIM lines

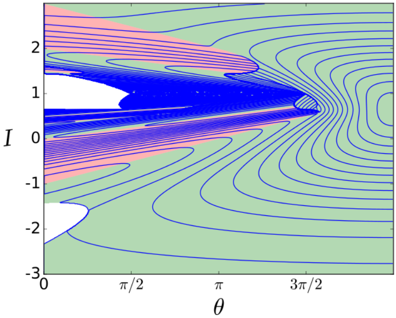


Figure: The “upper” crest

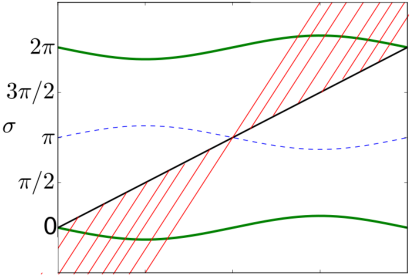


Figure: Minimal time

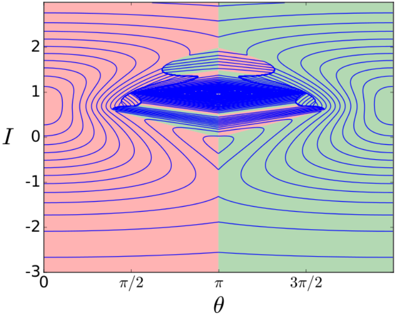


Figure: Minimal $|\tau^*|$ between "lower" and "upper" crest

In this picture we show a combination of 3 scattering maps.

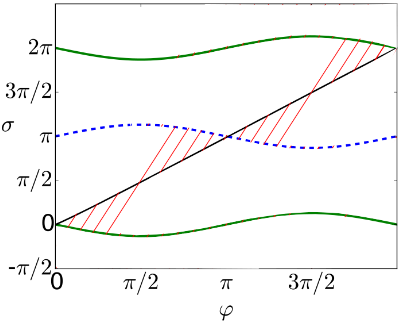


Figure: First intersection

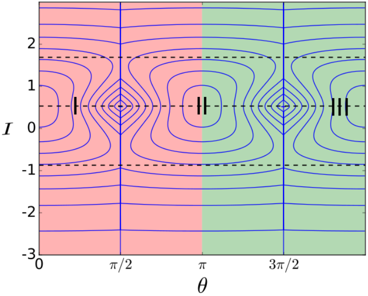


Figure: Minimal $|\tau^*|$ between $\mathcal{C}_{\mathrm{M}}(I)$ and $\mathcal{C}_{\mathrm{m}}(I)$

Consider a pendulum and two rotors plus a time periodic perturbation depending on three harmonics in the angles $(\varphi_1, \varphi_2, \varphi_3 = s)$:

$$H_\varepsilon(p,q,I_1,I_2,\varphi_1,\varphi_2,s) = \pm\left(\frac{p^2}{2}+\cos q - 1\right) + h(I_1,I_2) + \varepsilon f(q)\, g(\varphi_1,\varphi_2,s), \qquad (15)$$

$$h(I_1,I_2) = \Omega_1 I_1^2/2 + \Omega_2 I_2^2/2, \qquad f(q) = \cos q$$
$$g(\varphi_1,\varphi_2,s) = a_1\cos\varphi_1 + a_2\cos\varphi_2 + a_3\cos s. \qquad (16)$$

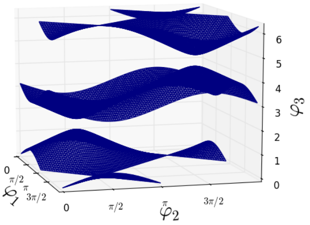
Theorem (Arnold diffusion for a two-parameter family)

Assume $a_1a_2a_3 \neq 0$ and $|a_1/a_3| + |a_2/a_3| < 0.625$ in Hamiltonian (15)+(16). Then, for any two actions $I_\pm$ and any δ there exists $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$ there exists an orbit $\tilde{x}(t)$ and $T > 0$ such that

$$|I(\tilde{x}(0)) - I_-| \le \delta \quad \text{and} \quad |I(\tilde{x}(T)) - I_+| \le \delta$$

For $|a_1/a_3| + |a_2/a_3| < 0.625$ there are two horizontal crests $\mathcal{C}_{\mathrm{M,m}}(I)$, and both scattering maps $\mathcal{S}_{\mathrm{M}}$, $\mathcal{S}_{\mathrm{m}}$ are globally well defined.

Figure: Horizontal crests: $a_1/a_3 = a_2/a_3 = 0.48$,$\Omega_1 I_1 = \Omega_2 I_2 = 1.219$.



Diffusing orbits are found by shadowing orbits of both scattering maps scattering maps and the inner dynamics.

Remark

Actually, we can prove that given any two actions $I_\pm$ and any path $\gamma(s)$ joining them in the actions space, there exists an orbit $\tilde{x}(t)$ such that $I(\tilde{x}(t))$ is δ-close to $\gamma(\Psi(t))$ for some parameterization Ψ.

Theorem (Diffusion paths using only Scattering maps)

Assume $a_1 a_2 a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$ *in Hamiltonian* (15)+(16). *Given any two* $(I_\pm, \theta_\pm) \in \tilde{\mathcal{I}}$, *where*

$$\tilde{\mathcal{I}} = \mathbb{R}^2 \times \mathbb{T}^2 \setminus \{(0,0,0,0),(0,0,\pi,0),(0,0,0,\pi),(0,0,\pi,\pi)\},$$

and any δ *there exists* $\varepsilon_0 > 0$ *such that for every* $0 < |\varepsilon| < \varepsilon_0$ *there is an orbit* $(I^i, \theta^i)_{0 \leq i < N}$ *of the polyscattering map* (S_0, S_1, S_2)*:*

$$(I^{i+1}, \theta^{i+1}) = S_\ell(I^i, \theta^i), \ \text{where } \ell \in \{0,1,2\},$$

such that

$$\left|(I^0, \theta^0) - (I_-, \theta_-)\right| < \delta \text{ and } \left|(I^N, \theta^N) - (I_+, \theta_+)\right| < \delta.$$

Theorem (Existence of Highways)

Assume $a_1 a_2 a_3 \neq 0$ *and* $|a_1/a_3| + |a_2/a_3| < 0.625$ *in Hamiltonian* (15)+(16). *Given any* $0 < c_j < C_j$, $j = 1, 2$, *there is an orbit* $(I^i, \theta^i)_{0 \leq i < N}$ *of the scattering map* S_0 *such that*

$$\left|I_j^0\right| < c_j \quad \text{and} \quad \left|I_j^N\right| > C_j, \qquad j = 1, 2.$$

(Quasi)-periodic perturbations of geodesic flows

Theorem ([D-Llave-Seara06])

Let M be a n-dimensional manifold, g a $\mathcal{C}^r$ metric on it (r sufficiently large). Assume:

- **H1** *There exists a closed geodesic "Λ" such that its corresponding periodic orbit $\hat{\Lambda}$ under the geodesic flow is hyperbolic.*
- **H2** *There exists another geodesic "γ" such that $\hat{\gamma}$ is a transversal homoclinic orbit to $\hat{\Lambda}$.*
  That is, $\hat{\gamma}$ is contained in the intersection of the stable and unstable manifolds of $\hat{\Lambda}$, $W^s_{\hat{\Lambda}}$, $W^u_{\hat{\Lambda}}$, in the unit tangent bundle.
  Moreover, we assume that the intersection of the stable and unstable manifolds of $\hat{\Lambda}$ is transversal along $\hat{\gamma}$. That is,

$$T_{\gamma(t)}W^s_{\hat{\Lambda}} + T_{\gamma(t)}W^u_{\hat{\Lambda}} = T_{\gamma(t)}\mathbf{S}_1 M, \quad t \in \mathbb{R}.$$

Abundance of Hypoteses **H1**, **H2**

Hipotheses **H1**, **H2** are abundant:

- They are generic on $\mathbb{T}^2$ [Morse24]] , [Hedlund32]] , [Mather93]] .
- They hold on any closed surface of genus bigger or equal than 2, if $r \geq 2 + \delta$, $\delta > 0$. [Katok82]).
- They are generic in the $\mathcal{C}^2$ topology for any closed surface [Contreras-Paternain02] .

(Quasi)-periodic perturbations of geodesic flows

Let $\nu \in \mathbb{R}^d$ be Diophantine, $r \in \mathbb{N}$ be sufficiently large (depending on τ, the Diophantine exponent of ν).
Let g be a $\mathcal{C}^r$ metric on a compact manifold M, verifying hypotheses **H1**, **H2**, and $U : M \times \mathbb{T}^d \to \mathbb{R}$ a generic $\mathcal{C}^r$ function.
Consider the time dependent Lagrangian

$$L(q, \dot{q}, \nu t) = \frac{1}{2} g^q(\dot{q}, \dot{q}) - U(q, \nu t), \tag{17}$$

where g^q denotes the metric in $\mathbf{T}_q M$.
Then, the Euler-Lagrange equation of L has a solution $q(t)$ whose energy

$$E(t) = \frac{1}{2} g^q(\dot{q}(t), \dot{q}(t)) + U(q(t), \nu t),$$

tends to infinity as $t \to \infty$.

The (planar) elliptic restricted three body problem (RPETB) describes the motion q of a massless particle (a comet) under the gravitational field of two massive bodies (the primaries, say the Sun and Jupiter) with mass ratio μ revolving around their center of mass on elliptic orbits with eccentricity ϵ_J.

Typical models:

- Sun–Jupiter–asteroid or comet: $\epsilon_J = 0.048$
- Sun–Earth–Moon systems: $\epsilon_J = 0.016$

We search for trajectories of motion which show a large variation of the angular momentum $G = q \times \dot{q}$.

So we search for global instability ("diffusion" is the term usually used) in the angular momentum of this problem.

Theorem (The Main Result)

There exist two constants $C > 0$, $c > 0$ *and* $\mu^* = \mu^*(C, c) > 0$ *such that for any* $0 < \epsilon_J < c/C$ *and* $0 < \mu < \mu^*$, *and for any two values of the angular momentum in the region* $C \leq G_1^* < G_2^* \leq c/\epsilon_J$, *there exists a trajectory of the RPETB such that* $G(0) < G_1^*$, $G(T) > G_2^*$ *for some* $T > 0$.

- If $\epsilon_J = 0$, the primaries revolve along circular orbits, and such diffusion is not possible, since the (planar) restricted circular three body problem (R3BP) is governed by an autonomous Hamiltonian with 2 degrees-of-freedom.
- This is not the case for the RPETB, which is a 2+1/2 degrees-of-freedom Hamiltonian system with time-periodic Hamiltonian.

Related results about oscillatory motions and diffusion for several Restricted Three Body Problems:

- Euler libration points: [Llibre-Martínez-Simó85, Capiński-Zgliczyński11, D-Gidea-Roldán13-16, Capiński-Llave-Gidea16, Kepley-Mireless James17]
- Collisions: [Bolotin06]
- The (parabolic) infinity: **[Llibre-Simó80]** , [Xia93-94, Moser01, Moeckel07] , **[Martínez-Pinyol94]** , [Gorodetski-Kaloshin11] , **[Guàrdia-Martín-Seara12]** , [Martínez-Simó14]
- Mean motion resonances: [Fejoz-Guàrdia-Kaloshin-Roldán14]
- Aubry-Mather theory: [Galante-Kaloshin13]

The motion of the massless particle q (comet) is described by

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_S - q}{|q_S - q|^3} + \mu\frac{q_J - q}{|q_J - q|^3}$$

where $1-\mu$ is the mass of the primary (Sun) at q_S and μ the mass of the primary (Jupiter) at q_J.

Introducing $p = dq/dt$, this is a 2+1/2 degree-of-freedom Hamiltonian system with time-periodic Hamiltonian

$$H_\mu(q,p,t;\epsilon_J) = \frac{p^2}{2} - U_\mu(q,t;\epsilon_J)$$

with self-potential

$$U_\mu(q,t;\epsilon_J) = \frac{1-\mu}{|q - q_S(t,\epsilon_J)|} + \frac{\mu}{|q - q_J(t,\epsilon_J)|}$$

Parameters: $0 < \mu, \epsilon_J < 1$ small.

When $\mu = 0$, there is no Jupiter in the equation of motion and the Sun is fixed at the origin: $q_S = 0$

The Sun q_S and the comet q form a two-body problem with the Hamiltonian $H_0(q, p, t; \epsilon_J) = H_0(q, p) = \dfrac{p^2}{2} - \dfrac{1}{|q|} = \dfrac{p^2}{2} - U_0(q)$.

The two–body problem is integrable, and there is no dependence on the eccentricity ϵ_J or the time t.

$$q_S = q_S(t, \epsilon_J) = \mu r(\cos f, \sin f)$$

$$q_J = q_J(t, \epsilon_J) = -(1-\mu) r(\cos f, \sin f)$$

with

$$r = r(t; \epsilon_J) = \frac{1-\epsilon_J^2}{1+\epsilon_J \cos f}, \qquad \frac{df}{dt} = \frac{(1+\epsilon_J \cos f)^2}{(1-\epsilon_J^2)^{3/2}},$$

where $f = f(t; \epsilon_J)$ is the true anomaly. If $q = \rho(\cos\alpha, \sin\alpha)$,

$$\begin{aligned} |q-q_S|^2 &= \rho^2 - 2\mu r\rho\cos(\alpha - f) + \mu^2 r^2, \\ |q-q_J|^2 &= \rho^2 + 2(1-\mu) r\rho\cos(\alpha - f) + (1-\mu)^2 r^2. \end{aligned}$$

Remark Also

$$r = r(t; \epsilon_J) = 1 - \epsilon_J \cos E, \qquad t = E - \epsilon_J \sin E,$$

where E is the eccentric anomaly.

Performing a standard polar-canonical change of variables $(q, p) \longmapsto (\rho, \alpha, P_\rho, P_\alpha)$

$$q = (\rho\cos\alpha, \rho\sin\alpha), \quad p = \left(P_\rho\cos\alpha - \frac{P_\alpha}{\rho}\sin\alpha, P_\rho\sin\alpha + \frac{P_\alpha}{\rho}\cos\alpha\right)$$

the Hamiltonian becomes

$$H^*_\mu(\rho, \alpha, P_\rho, P_\alpha, t; \epsilon_\mathrm{J}) = \frac{P_\rho^2}{2} + \frac{P_\alpha^2}{2\rho^2} - U^*_\mu(\rho, \alpha, t; \epsilon_\mathrm{J})$$

with a self-potential U^*_μ

$$U^*_\mu(\rho, \alpha, t; \epsilon_\mathrm{J}) = U_\mu(\rho\cos\alpha, \rho\sin\alpha, t; \epsilon_\mathrm{J}) = \frac{1}{\rho} + O(\mu).$$

From now on we will write

$$G = P_\alpha, \qquad y = P_\rho,$$

so that Hamiltonian (80) becomes

$$H^*_\mu(\rho, \alpha, y, G, t; \epsilon_J) = \frac{y^2}{2} + \frac{G^2}{2\rho^2} - U^*_\mu(\rho, \alpha, t; \epsilon_J).$$

Remark

In the (planar) circular case $\epsilon_J = 0$ (RTBP), $r = 1$ and $f = t$, and $|q - q_S|, |q - q_J|$ depend on the time t and the angle α just through their difference $\alpha - t$. As a consequence, $U^*_\mu(\rho, \alpha, t; 0)$ as well as $H^*_\mu(\rho, \alpha, y, G, t; 0)$ depend also on t and α just through the same difference $\alpha - t$, the sinodic angle. This implies that the Jacobi constant $H^* + G$ is a first integral of the system.

Through McGehee non-canonical change of variables, for $x > 0$,

$$\rho = \frac{2}{x^2}$$

the infinity $\rho = \infty$ is sent to the origin $x = 0$ and the equations become

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}_\mu}{\partial x}$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = \frac{\partial \mathcal{U}_\mu}{\partial \alpha},$$

where the self-potential $\mathcal{U}_\mu$ is given now by

$$\mathcal{U}_\mu(x, \alpha, t; \epsilon_J) = U^*_\mu(2/x^2, \alpha, t; \epsilon_J) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right)$$

with

$$|q - q_S|^2 = \sigma_S^2 = 1 - \mu r x^2 \cos(\alpha - f) + \frac{1}{4}\mu^2 r^2 x^4,$$

$$|q - q_J|^2 = \sigma_J^2 = 1 + (1-\mu) r x^2 \cos(\alpha - f) + \frac{1}{4}(1-\mu)^2 r^2 x^4.$$

Under McGehee change of variables $\rho = 2/x^2$ for $x > 0$,

$$d\rho \wedge dy + d\alpha \wedge dG \quad \text{is transformed to} \quad \omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$$

which is a b^3-symplectic form, the new Hamiltonian reads as

$$\mathcal{H}_\mu(x, \alpha, y, G, t; \epsilon_J) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_\mu(x, \alpha, t; \epsilon_J),$$

and the the Hamiltonian equations become

$$\frac{dx}{dt} = -\frac{x^3}{4}\left(\frac{\partial \mathcal{H}_\mu}{\partial y}\right) \qquad \frac{dy}{dt} = -\frac{x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right)$$

$$\frac{d\alpha}{dt} = \frac{\partial \mathcal{H}_\mu}{\partial G} \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{H}_\mu}{\partial \alpha}.$$

which can be written as $dz/dt = \{z, \mathcal{H}_\mu\}$ in terms of the Poisson bracket

$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial g}{\partial \alpha}.$$

- Some sources of b^m-symplectic structures can be found in [Scott13, Kiesenhofer-Miranda-Scott15, Guillemin-Miranda-Weitsman17-18] .
- Other examples can be found in [Guardia-Martín-Seara16, D-Kiesenhofer-Miranda17, Braddell-D-Miranda-Oms-Planas17] .
- New examples in [Baldomá-Fontich-Martín18] .

For $\mu = 0$ and $G > 0$, Hamiltonian $\mathcal{H}_0$ becomes Duffing Hamiltonian:

$$\mathcal{H}_0(x, y, G) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_0(x) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2}$$

$\mathcal{H}_0$ is autonomous and independent of ϵ_J and α. Its associated equations are

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = 0$$

The angular momentum G is a conserved quantity, $G > 0$ from now on. The phase space $(x, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+$ includes the set of equilibrium points

$$\mathcal{E}_\infty = \{z = (x = 0, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+\}.$$

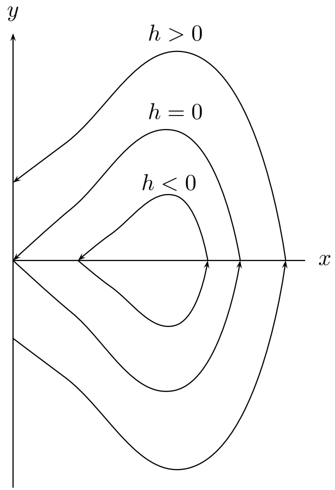


Figure: Level curves of $\mathcal{H}_0$ in the $(x \geq 0, y)$ plane, for fixed $G > 0$

For any fixed $\alpha \in \mathbb{T}$, $G \in \mathbb{R}$,

$$\Lambda_{\alpha,G} = \{(0, \alpha, 0, G)\}$$

is a parabolic equilibrium point, which is topologically equivalent to a saddle point, since it possesses stable and unstable 1D-invariant manifolds. The union of such points is the 2D-(symplectic) manifold of equilibrium points

$$\Lambda_\infty = \bigcup_{\alpha,G} \Lambda_{\alpha,G}.$$

which is the (parabolic) infinity manifold for the Kepler problem.
As we will deal with a time-periodic Hamiltonian, it is natural to work in the extended phase space

$$\tilde{z} = (z, s) = (x, \alpha, y, G, s) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$$

just by writing s instead of t in the Hamiltonian and adding the equation

$$\frac{ds}{dt} = 1$$

The extended versions of the invariant sets $\Lambda_{\alpha,G}$, Λ_∞ for the Kepler problem are the 2π-periodic orbits with motion $ds/dt = 1$

$$\tilde{\Lambda}_{\alpha,G} = \{\tilde{z} = (0, \alpha, 0, G, s), s \in \mathbb{T}\},$$

and the 3D-invariant manifold (the "parabolic" infinity manifold)

$$\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G} = \{(0, \alpha, 0, G, s),\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\}, \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},$$

which is topologically equivalent to a normally hyperbolic invariant manifold (TNHIM).
Parameterizing the points in $\tilde{\Lambda}_\infty$ by

$$\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0(\alpha, G, s) = (\mathbf{x}_0(\alpha, G), s) = (0, \alpha, 0, G, s) \in \tilde{\Lambda}_\infty \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$$

the inner dynamics on $\tilde{\Lambda}_\infty$ is trivial, since it is given by the dynamics on each periodic orbit $\tilde{\Lambda}_{\alpha,G}$:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t),$$

where we denote by $\tilde{\phi}_{t,\mu}$ the flow of our system in the extended phase space.

The equilibrium points $\Lambda_{\alpha,G}$ have stable and unstable 1D-invariant manifolds which coincide:

$$\begin{aligned}\gamma_{\alpha,G} &= W^{\mathrm{u}}(\Lambda_{\alpha,G}) = W^{\mathrm{s}}(\Lambda_{\alpha,G}) \\ &= \left\{ z = (x, \hat{\alpha}, y, G),\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx \right\},\end{aligned}$$

whereas the 2D-manifold of equilibrium points Λ_∞ has stable and unstable 3D-invariant manifolds which coincide and are given by

$$\gamma = W^{\mathrm{u}}(\Lambda_\infty) = W^{\mathrm{s}}(\Lambda_\infty) = \{z = (x, \alpha, y, G),\ \mathcal{H}_0(x, y, G) = 0\}.$$

In the extended phase space, the surface

$$\begin{aligned}\tilde{\gamma}_{\alpha,G} &= W^{\mathrm{u}}(\tilde{\Lambda}_{\alpha,G}) = W^{\mathrm{s}}(\tilde{\Lambda}_{\alpha,G}) \\ &= \left\{\tilde{z} = (x, \hat{\alpha}, y, G, s), s \in \mathbb{T},\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G\int_{\mathcal{H}_0=0} \frac{x}{y} dx\right\}\end{aligned}$$

is a 2D-homoclinic manifold to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$. The 4D-stable and unstable manifolds of the infinity manifold $\tilde{\Lambda}_\infty$ coincide along the 4D-homoclinic invariant manifold (the separatrix), which is just the union of the homoclinic surfaces $\tilde{\gamma}_{\alpha,G}$:

$$\begin{aligned}\tilde{\gamma} &= W^{\mathrm{u}}(\tilde{\Lambda}_\infty) = W^{\mathrm{s}}(\tilde{\Lambda}_\infty) = \bigcup_{\alpha,G} \tilde{\gamma}_{\alpha,G} \\ &= \{\tilde{z} = (x, \alpha, y, G, s), (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},\ \mathcal{H}_0(x, \alpha, y, G) = 0\}\end{aligned}$$

The homoclinic solutions to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ are given by

$$x_\mathrm{h}(t;G) = \frac{2}{G(1+\tau^2)^{1/2}} \qquad y_\mathrm{h}(t;G) = \frac{2\tau}{G(1+\tau^2)}$$
$$\alpha_\mathrm{h}(t;\alpha,G) = \alpha + \pi + 2\arctan\tau \qquad G_\mathrm{h}(t;G) = G$$
$$s_\mathrm{h}(t;s) = s + t \qquad ,$$

where α and G are the 2 free parameters and the relation between t and τ is

$$t = \frac{G^3}{2}\left(\tau + \frac{\tau^3}{3}\right) \quad \text{which is equivalent to} \quad \frac{dt}{d\tau} = \frac{2G}{x^2},$$

Due to the factor $-x^3/4$ in front of the equations, the convergence along the separatrix to the infinity manifold is power-like in τ and t:

$$x_\mathrm{h}, y_\mathrm{h}, \frac{\alpha - \alpha_\mathrm{h} + \pi}{G} \sim \frac{2}{G\tau} \sim \frac{2}{\sqrt[3]{\pm 6t}}, \quad \tau, t \to \pm\infty.$$

Introducing the notation

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s) = (x_{\mathrm{h}}(\sigma; G), \alpha_{\mathrm{h}}(\sigma; \alpha, G), y_{\mathrm{h}}(\sigma; G), G, s)$$

we can parameterize any homoclinic surface $\tilde{\gamma}_{\alpha,G}$ as

$$\tilde{\gamma}_{\alpha,G} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s),\ \sigma \in \mathbb{R}, s \in \mathbb{T}\}.$$

and the 4-dimensional separatrix $\tilde{\gamma} = W(\tilde{\Lambda}_\infty)$ as

$$\tilde{\gamma} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s), \sigma \in \mathbb{R}, G \in \mathbb{R}_+, (\alpha, s) \in \mathbb{T}^2\}.$$

The motion on $\tilde{\gamma}$ and $\tilde{\Lambda}_\infty$ is given by

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = \tilde{\mathbf{z}}_0(\sigma + t, \alpha, G, s + t) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t)$$
$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t),$$

and the following asymptotic formula follows:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) - \tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) - (\mathbf{x}_0(\alpha, G), s + t) \xrightarrow[t \to \pm\infty]{} 0.$$

The scattering map $\widetilde{S}$ describes the homoclinic orbits to the infinity manifold $\tilde{\Lambda}_\infty$. Given $\tilde{\mathbf{x}}_-, \tilde{\mathbf{x}}_+ \in \tilde{\Lambda}_\infty$, we define

$$\widetilde{S}_\mu(\tilde{\mathbf{x}}_-) := \tilde{\mathbf{x}}_+$$

if there exists $\tilde{\mathbf{z}}^* \in W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty) \cap W^s_\mu(\tilde{\Lambda}_\infty)$ such that

$$\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_\pm) \longrightarrow 0 \quad \text{for } t \to \pm\infty.$$

In the case $\mu = 0$ the previous asymptotic relation

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) - \tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) - (\mathbf{x}_0(\alpha, G), s + t) \xrightarrow[t \to \pm\infty]{} 0.$$

implies $\widetilde{S}_0(\tilde{\mathbf{x}}_0) = \tilde{\mathbf{x}}_0$ so that that the scattering map $\widetilde{S}_0 : \tilde{\Lambda}_\infty \longrightarrow \tilde{\Lambda}_\infty$ is the identity.

For $\mu > 0$, the set $\mathcal{E}_\infty$ remains invariant as well as infinity manifold $\tilde{\Lambda}_\infty$, which is again a TNHIM, as well as all the periodic orbits $\tilde{\Lambda}_{\alpha,G}$.

The inner dynamics on $\tilde{\Lambda}_\infty$ is the same as in the case $\mu = 0$, so that the parametrization $\tilde{\mathbf{x}}_0$ as well as its trivial dynamics remain the same.

From [McGehee73, Guardia-Martín-Seara-Sabbagh17] we know that $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ exist for μ small enough and are 4-dimensional in the extended phase space.

The existence of scattering maps will depend on the transversal intersections between these two manifolds.

Introduce now [D-Gutiérrez00, D-Llave-Seara06] the Melnikov potential $\mathcal{L} : \tilde{\Lambda}_\infty \longrightarrow \mathbb{R}$ by

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \int_{-\infty}^{\infty} \Delta\mathcal{U}_0(x_h(t; G), \alpha_h(t; \alpha, G), s + t; \epsilon_J)\, dt,$$

where $\Delta\mathcal{U}_0$ is defined by

$$\Delta\mathcal{U}_0(x, \alpha, s; \epsilon_J) := \left.\frac{\partial \mathcal{U}\mu}{\partial \mu}\right|_{\mu=0} (x, \alpha, s; \epsilon_J) = O(x^4) \quad \text{as} \quad x \to 0.$$

The asymptotics above follows from the asymptotic behavior of the solutions along the separatrix and of the self potential close to the parabolic infinity manifold, and guarantees that this integral is absolutely convergent.

Proposition (Transverse homoclinic points to the infinite manifold $\tilde\Lambda_\infty$)

Given $(\alpha, G, s) \in \mathbb{T} \times \mathbb{R}^+ \times \mathbb{T}$, *assume that the function*

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s-\sigma; \epsilon_J) \in \mathbb{R}$$

has a non-degenerate critical point $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$. *Then, there exists* $\mu^* = \mu^*(G, \epsilon_J)$, *such that for* $0 < \mu < \mu^*$, *close to the point* $\tilde{\mathbf{z}}_0^* = (\mathbf{z}_0(\sigma^*, \alpha, G), s) \in \tilde\gamma$ *there exists a locally unique point*

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*(\sigma^*, \alpha, G, s; \epsilon_J, \mu) \in W^s_\mu(\tilde\Lambda_\infty) \pitchfork W^u_\mu(\tilde\Lambda_\infty)$$

of the form $\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + O(\mu)$, *and there exist unique points* $\tilde{\mathbf{x}}_\pm = (0, \alpha_\pm, 0, G_\pm, s) = (0, \alpha, 0, G, s) + O(\mu) \in \tilde\Lambda_\infty$ *such that*

$$\tilde\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde\phi_{t,\mu}(\tilde{\mathbf{x}}_\pm) \longrightarrow 0 \quad \text{for } t \to \pm\infty.$$

Moreover, we have

$$G_+ - G_- = \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*(\alpha, G, s; \epsilon_J)) + O(\mu^2).$$

Once we have found a critical point $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$ of

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$$

on a domain of (α, G, s), we can define the reduced Poincaré function [D-Llave-Seara06]

$$\mathcal{L}^*(\alpha, G; \epsilon_J) := \mathcal{L}(\alpha, G, s - \sigma^*; \epsilon_J) = \mathcal{L}(\alpha, G, s^*; \epsilon_J)$$

with $s^* = s - \sigma^*$. Note that the reduced Poincaré function does not depend on the s chosen, since by the previous Proposition

$$\frac{\partial}{\partial s} \left(\mathcal{L}\left(\alpha, G, s - \sigma^*(\alpha, G, s; \epsilon_J); \epsilon_J\right)\right) \equiv 0.$$

Note also that if the function $\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$ has different non degenerate critical points there will exist different scattering maps.

The next Proposition gives an approximation of the scattering map in the general case $\mu > 0$.

Proposition (Expression of the scattering map)

The associated scattering map $(\alpha_+, G_+, s_+) = \widetilde{S}_\mu(\alpha, G, s)$ *for any non degenerate critical point* $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$ *of the function* $\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R}$ *is an exact symplectic map given by*

$$(\alpha, G, s) \longmapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G}(\alpha, G; \epsilon_J) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G; \epsilon_J) + O(\mu^2), s\right)$$

where $\mathcal{L}^*$ *is the Poincaré reduced function.*

Remark: the scattering map $\widetilde{S}_\mu$ follows closely the level curves of the Hamiltonians $\mathcal{L}^*$. More precisely, up to $O(\mu^2)$ terms, $\widetilde{S}_\mu$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}*$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| \ll \left|\frac{\partial \mathcal{L}^*}{\partial G}\right|, \left|\frac{\partial \mathcal{L}^*}{\partial \alpha}\right|.$$

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \int_{-\infty}^{\infty} \left[\frac{x_h^2}{\left[4 + x_h^4 r^2 + 4x_h^2 r \cos(\alpha_h - f)\right]^{1/2}} + \left(\frac{x_h^2}{2}\right)^2 r \cos(\alpha_h - f) - \frac{x_h^2}{2} \right] dt$$

where x_h and α_h, solutions on the separatrix, are evaluated at t, whereas r and f, concerning the primaries, are evaluated at $s + t$.

Fourier expanding with respect to angular variables α, s, $\mathcal{L}$ is an even function α, s: $\mathcal{L}(-\alpha, G, -s; \epsilon_J) = \mathcal{L}(\alpha, G, s; \epsilon_J)$, and therefore $\mathcal{L}$ has a Fourier Cosine series with real coefficients $L_{q,k}$:

$$\mathcal{L} = L_{0,0} + 2\sum_{k \geq 1} L_{0,k} \cos k\alpha + 2\sum_{q \geq 1}\sum_{k \in \mathbb{Z}} L_{q,k} \cos(qs + k\alpha).$$

Using the method of steepest descent along adequate complex paths, and playing both with the eccentric and the true anomaly, it is possible to compute these Fourier coefficients.

Theorem (Computation of the Melnikov potential)

For $G \geq 32$, $\epsilon_J G \leq 1/8$, the Melnikov potential is given by

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \mathcal{L}_0(\alpha, G; \epsilon_J) + \mathcal{L}_1(\alpha, G, s; \epsilon_J) + \mathcal{L}_{\geq 2}(\alpha, G, s; \epsilon_J)$$

with

$$\begin{aligned}\mathcal{L}_0(\alpha, G; \epsilon_J) &= L_{0,0} + L_{0,1}\cos\alpha + \mathcal{E}_0(\alpha, G; \epsilon_J)\\ \mathcal{L}_1(\alpha, G, s; \epsilon_J) &= 2L_{1,-1}\cos(s-\alpha) + 2L_{1,-2}\cos(s-2\alpha)\\ &\quad + 2L_{1,-3}\cos(s-3\alpha) + \mathcal{E}_1(\alpha, G, s; \epsilon_J),\end{aligned}$$

where $L_{i,j} = L_{i,j}(G; \epsilon_J)$ with $L_{0,0} = \frac{\pi}{2G^3}(1 + E_{0,0})$ and

$$L_{0,1} = -\frac{15\pi\epsilon_J}{8G^5}(1 + E_{0,1}), \qquad 2\,L_{1,-1} = \sqrt{\frac{\pi}{8G}}e^{-G^3/3}(1 + E_{1,-1})$$

$$2\,L_{1,-2} = -3\sqrt{2\pi}\epsilon_J G^{3/2}e^{-G^3/3}(1 + E_{1,-2})$$

$$2\,L_{1,-3} = \frac{19}{8}\sqrt{2\pi}\epsilon_J^2 G^{5/2}e^{-G^3/3}(1 + E_{1,-3}).$$

Theorem (Continuation of the computation of the Melnikov potential)

The error functions satisfy

$$
\begin{aligned}
|E_{0,0}| &\leq 2^{12} G^{-4} + 2^2\, 49\, \epsilon_J^2 \\
|E_{0,1}| &\leq 2^{13} G^{-4} + \epsilon_J^2 \\
|E_{1-1}| &\leq 2^{21} G^{-1} + 2\, 49\, \epsilon_J^2 \\
|E_{1,-2}| &\leq 2^{17} G^{-1} + \frac{49}{3}\epsilon_J \\
|E_{1,-3}| &\leq 2^{17} G^{-1} + 15\epsilon_J \\
|\mathcal{E}_0| &\leq 2^{14}\, \epsilon_J^2 G^{-7} \\
|\mathcal{E}_1| &\leq 2^{18} \epsilon_J e^{-G^3/3} \left[\epsilon_J^2 G^{7/2} + G^{-3/2}\right] \\
|\mathcal{L}_{\geq 2}| &\leq 2^{28} G^{3/2} e^{-2G^3/3}
\end{aligned}
$$

$s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is indeed a cosine-like function, that is, with a non-degenerate maximum (minimum) and no other critical points, so we can find easily its critical points.

Proposition

There exists $C > 32$ and $c < 1/8$ such that, for $G \geq C$ and $\epsilon_J G < c$, $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is a cosine-like function, and its two critical points are given by

$$s_+^* = s_+^*(\alpha, G; \epsilon_J) = \alpha + \theta + \varphi^*, \qquad s_-^* = s_-^* + \pi = \alpha + \theta + \pi + \varphi^*$$

where $\theta = \theta(\alpha, G; \epsilon_J)$ and $\varphi^* = O\left(G^{3/2} e^{-G^3/3}\right)$.

By the previous Theorem, for $G > C$ big enough and $G\epsilon_J < c$ small enough, the two critical points of $\mathcal{L}$ in the variable s are well approximated by the two critical points of the function $\mathcal{L}_0 + \mathcal{L}_1$ (in fact of $\mathcal{L}_1$ because $\mathcal{L}_0$ does not depend on s).

We can define two different reduced Poincaré functions

$$\begin{aligned}\mathcal{L}^*_\pm(\alpha, G; \epsilon_J) &= \mathcal{L}(\alpha, G, s^*_\pm; \epsilon_J) \\ &= \mathcal{L}_0(\alpha, G; \epsilon_J) \pm \mathcal{L}^*_1(\alpha, G; \epsilon_J) + \mathcal{E}_\pm(\alpha, G; \epsilon_J).\end{aligned}$$

and two different scattering maps $\widetilde{S}_\pm(\alpha, G, s) = (S_\pm(\alpha, G, s), s)$, where

$$S_\pm(\alpha, G, s) = \left(\alpha - \mu \frac{\partial \mathcal{L}^*_\pm}{\partial G}(\alpha, G; \epsilon_J) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*_\pm}{\partial \alpha}(\alpha, G; \epsilon_J) + O(\mu^2)\right).$$

which follow closely the level curves of the Hamiltonians $\mathcal{L}^*_\pm$. More precisely, up to $O(\mu^2)$ terms, $S_\pm$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}^*_\pm$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| \ll \left|\frac{\partial \mathcal{L}^*_\pm}{\partial G}\right|, \left|\frac{\partial \mathcal{L}^*_\pm}{\partial \alpha}\right|,$$

which is true as long as $0 < \mu \ll \mu^* = e^{-(c/\epsilon_J)^3/3}$.

One has to check that the foliations of $\mathcal{L}^*_\pm =$ constant are different, since this will imply that the scattering maps $S_\pm$ are different. From

$$\begin{aligned}\{\mathcal{L}^*_+, \mathcal{L}^*_-\} &= \{\mathcal{L}_0 + \mathcal{L}^*_1 + \cdots, \mathcal{L}_0 - \mathcal{L}^*_1 + \cdots\} \\ &= -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \mathcal{E}_3\end{aligned}$$

one computes

$$\{\mathcal{L}_0, \mathcal{L}^*_1\} = -\frac{15\pi\epsilon_J \mathcal{L}^*_1 d \sin\alpha}{8G^3B^2}.$$

The level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are transversal in the region $G \geq C > 32$ and $\epsilon_J G \leq c < 1/8$, except for the three curves $\alpha = 0$, $\alpha = \pi$ and $d = 0$, which are transversal to any of these level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$, see next slide.

Indeed, this is clear for the lines $\alpha = 0$ and $\alpha = \pi$, and the same happens for the curve $d = 0$ using its complete expression.

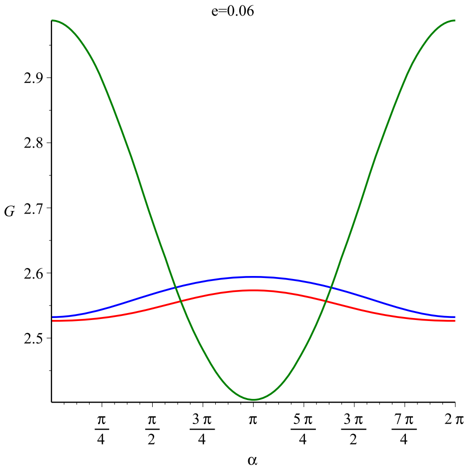


Figure: Illustration of the level Sets of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in Blue (Red) and $d = 0$ in Green

- Apart from these three curves $\alpha = 0, \pi$ and $d = 0$, at any point in the plane (α, G) the slopes $dG/d\alpha$ of the level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are different.
- We can choose which level curve increases more the value of G (see next slide).
- In the same way, we can find trajectories along which the angular momentum performs arbitrary excursions.
- Strictly speaking, this mechanism only produces pseudo-orbits, that is, heteroclinic connections between different periodic orbits in the infinity manifold which are commonly known as transition chains after Arnold.
- The existence of true orbits relies on shadowing methods [Moeckel02-07,Gidea-Llave06,Gidea-Llave-Seara14, Guardia-Martín-Seara-Sabbagh17] .

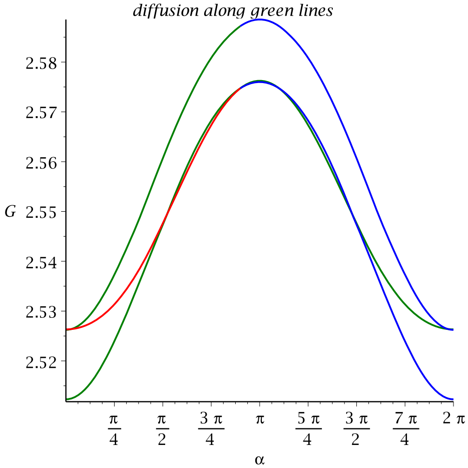


Figure: Zone of diffusion: Level curves of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in blue (red) and diffusion trajectories in green.

Theorem (Main Result again)

Let $G_1^* < G_2^*$ large enough and $\epsilon_J > 0$, $\mu > 0$ small enough. More precisely $C \leq G_1^* < G_2^* \leq c/\epsilon_J$ and $0 < \mu < \mu^* = \frac{c}{C} e^{-(8\epsilon_J)^{-3}/3}$, for $C < 32$ large enough and $c < 1/8$ small enough. Then, for any finite sequence of values $G_i \in (G_1^*, G_2^*)$, $i = 1, \ldots, n$, there exists a trajectory of the RPETB such that $G(T_i) = G_i$, $i = 1, \ldots, n$ for some $0 < T_i < T_{i+1}$. In particular, for any two values $G_1 < G_2 \in (G_1^*, G_2^*)$, there exists a trajectory such that $G(0) < G_1$, and $G(T) > G_2$ for some time $T > 0$.

Arnold's mechanism of diffusion in the spatial RTBP

- Model:
  - The spatial circular restricted three-body problem: an infinitesimal mass moves in space under the gravitational influence of two massive bodies (primaries) describing circular orbits, without exerting any influence on them
  - Focus on the dynamics near L_1, the libration point between the primaries – center$\times$center$\times$saddle
- Results:
  - There exist trajectories that change the out-of-plane amplitude (w.r. to the ecliptic) of an orbit near L_1 by a 'significant amount', via the Arnold mechanism of instability
    - abstract theorem – if certain conditions hold true then the existence of drift trajectories follows
    - verification of conditions – some analytical, some numerical
  - Related works [Samà04,Terra,Simó,de Sousa Silva14]

Introduction

- Method:
  - There exists a normally hyperbolic invariant three-sphere
  - We construct orbits that alternatively follow segments of homoclinic trajectories (outer dynamics) with segments of trajectories restricted to the three-sphere (inner dynamics), thus mimicking Arnold's instability mechanism of transition tori[1]
  - However, we use only coarse information on the inner dynamics (Poincaré recurrence theorem), no detailed information on the invariant objects (KAM tori, Aubry-Mather sets, etc.)
  - We use a geometric method that allows for explicit construction of drifting trajectories under milder conditions on the dynamics (compared to variational methods)
  - This is a general strategy

[1] Our model is not a small perturbation of an integrable system

Reference Problem: 3D Circular RTBP

The Restricted Three Body Problem (RTBP) defined as

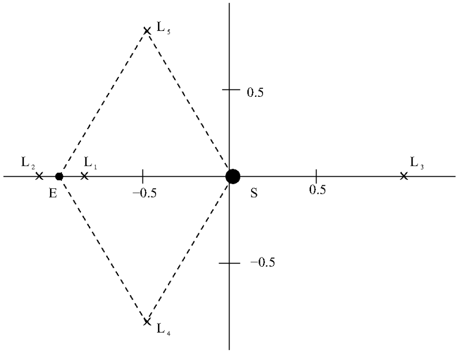
$$\begin{aligned} \ddot{X} - 2\dot{Y} &= \Omega_X, \\ \ddot{Y} + 2\dot{X} &= \Omega_Y, \\ \ddot{Z} &= \Omega_Z, \end{aligned}$$

where

$$\Omega = \frac{1}{2}(X^2 + Y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu),$$

$$\begin{aligned} r_1^2 &= (X-\mu)^2 + Y^2 + Z^2, \\ r_2^2 &= (X-\mu+1)^2 + Y^2 + Z^2. \end{aligned}$$

Libration Points



X-coordinate of L_1 is

$$X_1 = -1 + \left(\frac{\mu}{3}\right)^{1/3} - \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \mathrm{O}\,\mu.$$

In the Sun-Earth system,

Birkhoff Normal Form

On the center manifold, we obtain a two degrees of freedom Hamiltonian

$$H_c = H_N\left(0, \frac{x_2^2+y_2^2}{2}, \frac{x_3^2+y_3^2}{2}\right).$$

Define the action-angle coordinates

$$I_p := \frac{x_2^2+y_2^2}{2}, \qquad \phi_p$$
$$I_v := \frac{x_3^2+y_3^2}{2}, \qquad \phi_v.$$

- The equations of motion are integrable

$$\dot I_p = 0, \qquad \dot\phi_p = \frac{\partial H}{\partial I_p} = \omega_p(I_p, I_v) \tag{18}$$
$$\dot I_v = 0, \qquad \dot\phi_v = \frac{\partial H}{\partial I_v} = \omega_v(I_p, I_v), \tag{19}$$

  and each solution lies on a 2-dimensional torus.
- Each torus can be identified with the actions I_p, I_v.

Family of Invariant Tori

- Let us fix the energy level to $H(0, I_p, I_v) = h$, with $H(L_1) \le h \le H(halo)$.
- Then we obtain a one-parameter family of invariant tori, parametrized by the vertical action I_v.

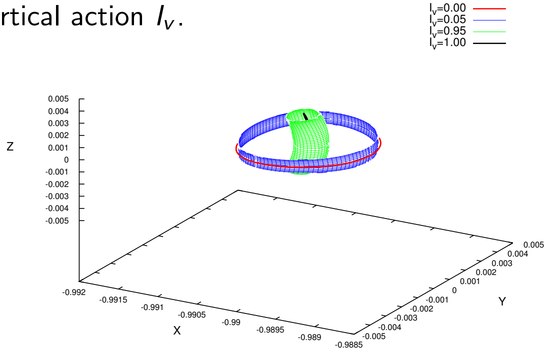


Figure: Low energy level $C = 3.00088$

Family of Invariant Tori

- Let us fix the energy level to $H(0, I_p, I_v) = h$, with $H(L_1) \leq h \leq H(halo)$.
- Then we obtain a one-parameter family of invariant tori, parametrized by the vertical action I_v.

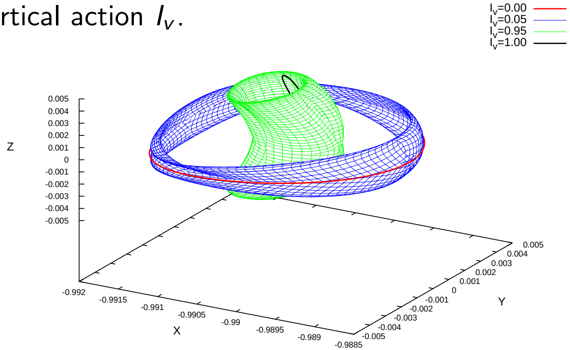


Figure: High energy level $C = 3.00083$

Transition Matrix

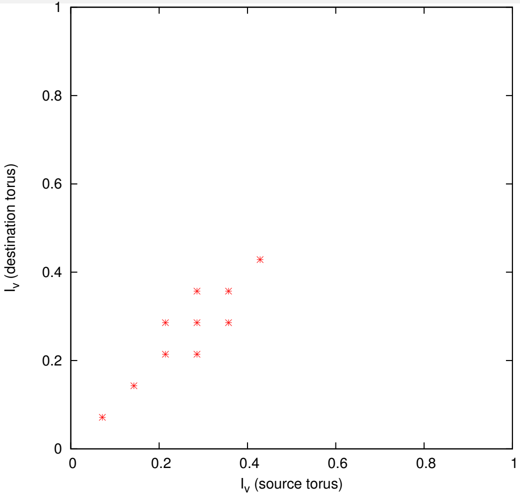


Figure: Low energy level $C = 3.00087$

Transition Matrix

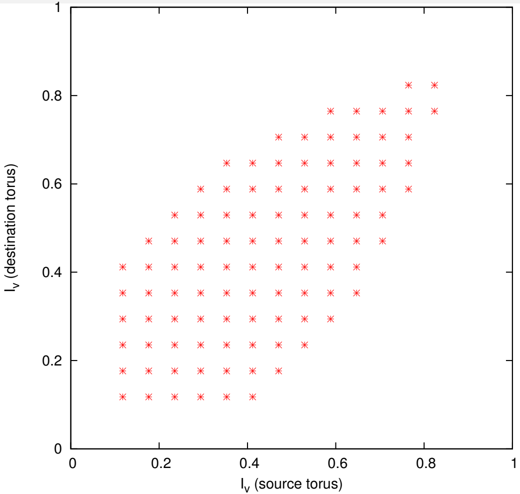


Figure: High energy level $C = 3.00083$

Main theoretical result (D-Gidea-Roldán 17)

Main Theorem. Given $\delta > 0$.
Assume $\exists \{\mathcal{L}^{\Sigma}_{I_j}\}_{j=0,N}$ level sets of I_v, with $0 < I_j < I_{max}$, and δ_j with $0 < \delta_j < \delta/2$, s.t., for each $j = 0, \dots, N-1$:

(i) $\exists$ scattering map $\sigma^{\Sigma}_{i(j)}$ and pt. $(I_j, \phi_j) \in \mathcal{L}^{\Sigma}_{I_j}$ s.t. $B_{\delta_j}(I_j, \phi_j) \subset \mathrm{dom}\sigma^{\Sigma}_{i(j)}$,

(ii) $\exists k_j > 0$ s.t. $\mathrm{int}[F^{k_j} \circ \sigma^{\Sigma}_{i(j)}(B_{\delta_j}(I_j, \phi_j))] \supseteq B_{\delta_{j+1}}(I_{j+1}, \phi_{j+1})$

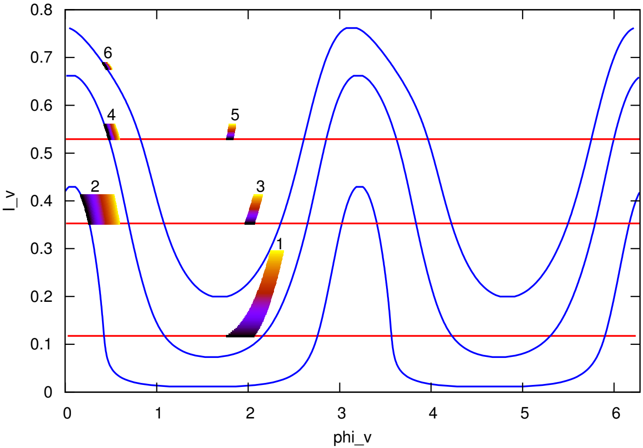
Then $\exists$ an orbit z_j of F in Σ, $j = 0, \dots, N$, and a sequence of positive integers $n_j > 0$, $j = 0, \dots, N-1$, such that $z_{j+1} = F^{n_j}(z_j)$ and

$$d(z_j, \mathcal{L}^{\Sigma}_{I_j}) < \delta/2, \text{ for all } j = 0, \dots, N. \tag{20}$$

Consequently, there exist a trajectory $\Phi^t(z)$ of the Hamiltonian flow, and a finite sequence of times $0 = t_0 < t_1 < t_2 < \dots < t_N$, such that

$$d(\Phi^{t_j}(z), \mathcal{L}_{I_j}) < \delta. \tag{21}$$

Main theoretical result



Future work

- Try to find drift orbits by constructing pseudo-orbits consisting of successive applications of several scattering maps
- Obtain theoretical results, using Hill locally and Kepler globally
- Add time dependent perturbation—elliptic orbit of Jupiter—and derive the existence of drift orbits

Bibliography

V. I. Arnold.

Small denominators and problems of stability of motion in classical and celestial mechanics.

Russian Math. Surveys.

V. Arnold.

Instability of dynamical systems with several degrees of freedom.

Sov. Math. Doklady, 5:581–585, 1964.

M. Berti, L. Biasco and P. Bolle.

Drift in phase space: a new variational mechanism with optimal diffusion time.

J. Math. Pures Appl. (9), 82(6):613–664, 2003.

U. Bessi, L. Chierchia and E. Valdinoci.

Upper bounds on Arnold diffusion times via Mather theory.

J. Math. Pures Appl. (9), 80(1):105–129, 2001.

R. Braddell, A. Delshams, E. Miranda, C. Oms and A. Planas.

An invitation to singular symplectic geometry, 2017.

To appear in *Int. J. Geom. Methods Mod. Phys.*

I. Baldomá and E. Fontich.
Stable manifolds associated to fixed points with linear part equal to identity.
J. Differential Equations, 197(1):45–72, 2004.

—.
Exponentially small splitting of separatrices in a weakly hyperbolic case.
J. Differential Equations, 210(1):106–134, 2005.

I. Baldomá, E. Fontich, M. Guardia and T. M. Seara.
Exponentially small splitting of separatrices beyond Melnikov analysis: rigorous results.
J. Differential Equations, 253(12):3304–3439, 2012.

P. Bernard, V. Kaloshin and K. Zhang.
Arnold diffusion in arbitrary degrees of freedom and normally hyperbolic invariant cylinders.
Acta Math., 217(1):1–79, 2016.

S. Bolotin.
Symbolic dynamics of almost collision orbits and skew products of symplectic maps.

Nonlinearity, 19(9):2041–2063, 2006.

G. Contreras-Barandiarán and G. P. Paternain.
Genericity of geodesic flows with positive topological entropy on S^2.
J. Differential Geom., 61(1):1–49, 2002.

L. Chierchia and G. Gallavotti.
Drift and diffusion in phase space.
Ann. Inst. H. Poincaré Phys. Théor., 60(1):144, 1994.

—.
Erratum: Drift and diffusion in phase space.
Ann. Inst. H. Poincaré Phys. Théor., 68:135, 1998.

M. J. Capiński, M. Gidea and R. de la Llave.
Arnold diffusion in the planar elliptic restricted three-body problem: mechanism and numerical verification.
Nonlinearity, 30(1):329–360, 2017.

C.-Q. Cheng.
Dynamics around the double resonance.
Camb. J. Math., 5(2):153–228, 2017.

C.-Q. Cheng and J. Yan.
Existence of diffusion orbits in a priori unstable Hamiltonian systems.
J. Differential Geom., 67(3):457–517, 2004.

—.
Arnold diffusion in Hamiltonian systems: a priori unstable case.
J. Differential Geom., 82(2):229–277, 2009.

M. J. Capiński and P. Zgliczyński.
Transition tori in the planar restricted elliptic three body problem, preprint.
Nonlinearity, 24(5):1395, 2011.

A. Delshams and P. Gutiérrez.
Effective stability and KAM theory.
J. Differential Equations, 128(2):415–490, 1996.

—.
Estimates on invariant tori near an elliptic equilibrium point of a Hamiltonian system.
J. Differential Equations, 131(2):277–303, 1996.

Bibliography

—.
Splitting potential and the Poincaré–Melnikov method for whiskered tori in Hamiltonian systems.
J. Nonlinear Sci., 10(4):433–476, 2000.

—.
Homoclinic orbits to invariant tori in Hamiltonian systems.
In C. Jones and A. Khibnik, editors, *Multiple-Time-Scale Dynamical Systems (Minneapolis, MN, 1997)*, volume 122 of *IMA Vol. Math. Appl.*, pages 1–27. Springer-Verlag, New York, 2001.

—.
Exponentially small splitting of separatrices for whiskered tori in Hamiltonian systems.
Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 300:87–121, 2003.
(*J. Math. Sci. (N.Y.)*, 128(2):2726–2746, 2005).

—.
Exponentially small splitting for whiskered tori in Hamiltonian systems: continuation of transverse homoclinic orbits.

Discrete Contin. Dyn. Syst., 11(4):757–783, 2004.

A. Delshams, M. Gonchenko and P. Gutiérrez.
Exponentially small asymptotic estimates for the splitting of separatrices to whiskered tori with quadratic and cubic frequencies.
Electron. Res. Announc. Math. Sci., 21:41–61, 2014.

—.
Exponentially small lower bounds for the splitting of separatrices to whiskered tori with frequencies of constant type.
Internat. J. Bifur. Chaos Appl. Sci. Engrg., 24(8):1440011, 12 pp., 2014.

—.
Continuation of the exponentially small transversality for the splitting of separatrices to a whiskered torus with silver ratio.
Regul. Chaotic Dyn., 19(6):663–680, 2014.

—.
A methodology for obtaining asymptotic estimates for the exponentially small splitting of separatrices to whiskered tori with quadratic frequencies.
In M. Corbera, J. Cors, J. Llibre and A. Korobeinikov, editors, *Extended Abstracts Spring 2014: Hamiltonian Systems and Celestial Mechanics; Virus Dynamics and*

Evolution, volume 4 of *Research Perspectives CRM Barcelona (Trends Math.)*, pages 31–37. Birkhäuser Basel, 2015.

—.
Exponentially small splitting of separatrices and transversality associated to whiskered tori with quadratic frequency ratio.
SIAM J. Appl. Dyn. Syst., 15(2):981–1024, 2016.

A. Delshams, V. Gelfreich, A. Jorba and T. M. Seara.
Exponentially small splitting of separatrices under fast quasiperiodic forcing.
Comm. Math. Phys., 189(1):35–71, 1997.

A. Delshams, M. Gidea and P. Roldan.
Transition map and shadowing lemma for normally hyperbolic invariant manifolds.
Discrete and Continuous Dynamical Systems. Series A., 3(33):1089–1112, 2013.

—.
Arnold's mechanism of diffusion in the spatial circular restricted three-body problem: A semi-analytical argument.
Physica D: Nonlinear Phenomena, 334(Supplement C):29 – 48, 2016.
Topology in Dynamics, Differential Equations, and Data.

Bibliography

A. Delshams, P. Gutiérrez and T. Seara.
Exponentially small splitting for whiskered tori in Hamiltonian systems: flow-box coordinates and upper bounds.
Discrete Contin. Dyn. Syst., 11(4):785–826, 2004.

A. Delshams and G. Huguet.
Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems.
Nonlinearity, 22(8):1997–2077, 2009.

—.
A geometric mechanism of diffusion: rigorous verification in a priori unstable Hamiltonian systems.
J. Differential Equations, 250(5):2601–2623, 2011.

A. Delshams, A. Kiesenhofer and E. Miranda.
Examples of integrable and non-integrable systems on singular symplectic manifolds.
J. Geom. Phys., 115:89–97, 2017.

A. Delshams, R. de la Llave and T. M. Seara.

A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model.
Mem. Amer. Math. Soc., 179(844), 2006.

—.
Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows.
Adv. Math., 202(1):64–188, 2006.

—.
Geometric properties of the scattering map of a normally hyperbolic invariant manifold.
Adv. Math., 217(3):1096–1153, 2008.

—.
Instability of high dimensional Hamiltonian systems: multiple resonances do not impede diffusion.
Adv. Math., 294:689–755, 2016.

A. Delshams and R. Ramírez-Ros.
Exponentially small splitting of separatrices for perturbed integrable standard-like maps.
J. Nonlinear Sci., 8(3):317–352, 1998.

Bibliography

—.
Singular separatrix splitting and the Melnikov method: an experimental study.
Experiment. Math., 8(1):29–48, 1999.

A. Delshams and T. M. Seara.
An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum.
Comm. Math. Phys., 150(3):433–463, 1992.

A. Delshams and R. G. Schaefer.
Arnold diffusion for a complete family of perturbations.
Regular and Chaotic Dynamics, 22(1):78–108, 2017.

—.
Arnold diffusion for a complete family of perturbations with two independent harmonics, 2017.
Preprint, arXiv:1710.00029.

M. N. Davletshin and D. V. Treschev.
Arnold diffusion in a neighborhood of strong resonances.
Proc. Steklov Inst. Math., 295(1):63–94, 2016.

Bibliography

L. H. Eliasson.
Biasymptotic solutions of perturbed integrable Hamiltonian systems.
Bol. Soc. Brasil. Mat. (N.S.), 25(1):57–76, 1994.

J. Féjoz, M. Guàrdia, V. Kaloshin and P. Roldán.
Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem.
J. Eur. Math. Soc. (JEMS), 18(10):2315–2403, 2016.

E. Fontich and P. Martín.
Differentiable invariant manifolds for partially hyperbolic tori and a lambda lemma.
Nonlinearity, 13(5):1561–1593, 2000.

E. Fontich.
Exponentially small upper bounds for the splitting of separatrices for high frequency periodic perturbations.
Nonlinear Anal., 20(6):733–744, 1993.

—.
Rapidly forced planar vector fields and splitting of separatrices.
J. Differential Equations, 119(2):310–335, 1995.

Bibliography

E. Fontich and C. Simó.
The splitting of separatrices for analytic diffeomorphisms.
Ergodic Theory Dynam. Systems, 10(2):295–318, 1990.

V. G. Gelfreich.
Separatrices splitting for the rapidly forced pendulum.
In *Seminar on Dynamical Systems (St. Petersburg, 1991)*, volume 12 of *Progr. Nonlinear Differential Equations Appl.*, pages 47–67. Birkhäuser, Basel, 1994.

J. Galante and V. Kaloshin.
Destruction of invariant curves in the restricted circular planar three-body problem by using comparison of action.
Duke Math. J., 159(2):275–327, 2011.

A. Gorodetski and V. Kaloshin.
On Hausdorff dimension of oscillatory motions in three body problems, 2013.
Preprint.

M. Gidea and R. de la Llave.
Topological methods in the instability problem of Hamiltonian systems.
Discrete Contin. Dyn. Syst., 14(2):295–328, 2006.

Bibliography

M. Gidea, R. de la Llave and T. M. Seara.
A general mechanism of diffusion in Hamiltonian systems: qualitative results.
arXiv preprint, pages 1–33, 2014.

M. Gidea and J.-P. Marco.
Diffusion along chains of normally hyperbolic cylinders, 2017.
Preprint, arXiv:1708.08314.

M. Guardia, P. Martín and T. M-Seara.
Oscillatory motions for the restricted planar circular three body problem.
Invent. Math., 203(2):417–492, 2016.

M. Guardia, P. Martín, T. M-Seara and L. Sabbagh.
Oscillatory orbits in the restricted elliptic planar three body problem.
Discrete Contin. Dyn. Syst., 37(1):229–256, 2017.

M. Guardia, C. Olivé and T. M. Seara.
Exponentially small splitting for the pendulum: a classical problem revisited.
J. Nonlinear Sci., 20(5):595–685, 2010.

V. Gelfreich, C. Simó and A. Vieiro.

Dynamics of $4D$ symplectic maps near a double resonance.
Phys. D, 243:92–110, 2013.

V. Gelfreich and D. Turaev.
Arnold diffusion in a priori chaotic symplectic maps.
Comm. Math. Phys., 353(2):507–547, 2017.

G. A. Hedlund.
Geodesics on a two-dimensional Riemannian manifold with periodic coefficients.
Ann. of Math., 33:719–739, 1932.

A. Katok.
Entropy and closed geodesics.
Ergodic Theory Dynam. Systems, 2(3-4):339–365 (1983), 1982.

V. Kaloshin, M. Levi and M. Saprykina.
Arnol'd diffusion in a pendulum lattice.
Comm. Pure Appl. Math., 67(5):748–775, 2014.

S. Kepley and J. Mireles James.
Chaotic motions in the restricted four body problem via devaney's saddle-focus homoclinic tangle theorem.

arXiv preprint, pages 1–67, 2017.

A. Kiesenhofer, E. Miranda and G. Scott.
Action-angle variables and a KAM theorem for *b*-Poisson manifolds.
J. Math. Pures Appl. (9), 105(1):66–85, 2016.

V. Kaloshin and K. Zhang.
Arnold diffusion for smooth convex systems of two and a half degrees of freedom.
Nonlinearity, 28(8):2699–2720, 2015.

J. Llibre, R. Martínez and C. Simó.
Tranversality of the invariant manifolds associated to the Lyapunov family of periodic orbits near L_2 in the restricted three-body problem.
J. Differential Equations, 58(1):104–156, 1985.

P. Lochak, J.-P. Marco and D. Sauzin.
On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems.
Mem. Amer. Math. Soc., 163(775), 2003.

L. Lazzarini, J.-P. Marco and D. Sauzin.

Measure and capacity of wandering domains in gevrey near-integrable exact symplectic systems, 2016.
Preprint, arXiv:1507.02050. To appear in *Mem. Amer. Math. Soc.*

P. Lochak.
Canonical perturbation theory via simultaneous approximation.
Russian Math. Surveys, 47(6):57–133, 1992.

J. Llibre and C. Simó.
Oscillatory solutions in the planar restricted three-body problem.
Math. Ann., 248(2):153–184, 1980.

J.-P. Marco.
Arnold diffusion for cusp-generic nearly integrable convex systems on $\mathbb{A}^3$, 2016.
Preprint, arXiv:1602.02403.

J. Mather.
Variational construction of connecting orbits.
Ann. Inst. Fourier (Grenoble), 43(5):1349–1386, 1993.

J. N. Mather.

Arnol′d diffusion. I. Announcement of results.
J. Math. Sci. (N. Y.), 124(5):5275–5289, 2004.

R. McGehee.
A stable manifold theorem for degenerate fixed points with applications to celestial mechanics.
J. Differential Equations, 14:70–88, 1973.

P. Martí n, D. Sauzin and T. M. Seara.
Exponentially small splitting of separatrices in the perturbed McMillan map.
Discrete Contin. Dyn. Syst., 31(2):301–372, 2011.

R. Moeckel.
Generic drift on Cantor sets of annuli.
In *Celestial mechanics (Evanston, IL, 1999)*, volume 292 of *Contemp. Math.*, pages 163–171. Amer. Math. Soc., Providence, RI, 2002.

—.
Symbolic dynamics in the planar three-body problem.
Regul. Chaotic Dyn., 12(5):449–475, 2007.

M. Morse.

A fundamental class of geodesics on any closed surface of genus greater than one.
Trans. Amer. Math. Soc., 26:26–60, 1924.

J. Moser.
Stable and random motions in dynamical systems.
Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.
ISBN 0-691-08910-8.
With special emphasis on celestial mechanics, Reprint of the 1973 original, With a foreword by Philip J. Holmes.

R. Martínez and C. Pinyol.
Parabolic orbits in the elliptic restricted three body problem.
J. Differential Equations, 111(2):299–339, 1994.

J.-P. Marco and D. Sauzin.
Stability and instability for Gevrey quasi-convex near-integrable Hamiltonian systems.
Publ. Math. Inst. Hautes Études Sci., (96):199–275 (2003), 2002.

J.-P. Marco and L. Sabbagh.

Examples of nearly integrable systems on $\mathbb{A}^3$ with asymptotically dense projected orbits, 2014.
ArXiv:1401.3593.

R. Martínez and C. Simó.
Invariant manifolds at infinity of the RTBP and the boundaries of bounded motion.
Regul. Chaotic Dyn., 19(6):745–765, 2014.

N. N. Nekhoroshev.
An exponential estimate of the time of stability of nearly integrable Hamiltonian systems.
Russian Math. Surveys, 32(6):1–65, 1977.

J. Pöschel.
Nekhoroshev estimates for quasi-convex Hamiltonian systems.
Math. Z., 213(2):187–216, 1993.

G. N. Piftankin and D. V. Treshchëv.
Separatrix maps in Hamiltonian systems.
Russian Math. Surveys, 62(2):219–322, 2007.

D. Sauzin.

Résurgence paramétrique et exponentielle petitesse de l'écart des séparatrices du pendule rapidement forcé.
Ann. Inst. Fourier (Grenoble), 45(2):453–511, 1995.

C. Simó.
Averaging under fast quasiperiodic forcing.
In *Hamiltonian mechanics (Toruń, 1993)*, volume 331 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 13–34. Plenum, New York, 1994.

C. Simó and A. Vieiro.
Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.
Nonlinearity, 22(5):1191–1245, 2009.

D. V. Treschev.
Splitting of separatrices for a pendulum with rapidly oscillating suspension point.
Russian J. Math. Phys., 5(1):63–98 (1998), 1997.

D. Treschev.
Arnold diffusion far from strong resonances in multidimensional *a priori* unstable Hamiltonian systems.

Nonlinearity, 25(9):2717–2757, 2012.

Z. Xia.
Arnol′d diffusion in the elliptic restricted three-body problem.
J. Dynam. Differential Equations, 5(2):219–240, 1993.

—.
Arnol′d diffusion and oscillatory solutions in the planar three-body problem.
J. Differential Equations, 110(2):289–321, 1994.

K. Zhang.
Speed of Arnold diffusion for analytic Hamiltonian systems.
Invent. Math., 186(2):255–290, 2011.